Compatibly split subvarieties of $\mathrm{Hilb}_n(\mathbb{A}_k^2)$

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April 1-2, 2011

Outline

Part 1: Compatibly split subschemes of $\mathrm{Hilb}_n(\mathbb{A}^k_2)$

- A brief introduction to $\mathrm{Hilb}_n(\mathbb{A}^k_2)$
- Stratifying $\mathrm{Hilb}_2(\mathbb{A}^k_2)$ by reduced subschemes
- A little bit of Frobenius splitting and an algorithm of Knutson, Lam and Speyer
- Compatibly split subschemes of $\mathrm{Hilb}_n(\mathbb{A}^k_2)$ for $n = 2, 3, 4$

Part 2: Restricting to an open affine patch

- A description of the affine patch U_λ, $\lambda = \langle x, y^n \rangle$
- Compatibly split subschemes of U_λ
- Gröbner degeneration and some combinatorics

Note: For the purposes of this talk, let k be an algebraically closed field of characteristic p.

$\mathrm{Hilb}_n(\mathbb{A}_k^2)$: Some Basics

Definition: As a set, $\mathrm{Hilb}_n(\mathbb{A}_k^2)$ is:

$$\{I \subset k[x,y] : \dim(k[x,y]/I) = n \text{ as a vector space over } k\}.$$

Each element $I \in \mathrm{Hilb}_n(\mathbb{A}_k^2)$ corresponds to "n points in the affine plane".

$\mathrm{Hilb}_n(\mathbb{A}_k^2)$: Some Basics (continued)

More precisely, the following is true:

- If I is a radical colength n ideal, then $\mathrm{Spec}(k[x,y]/I)$ consists of n distinct points in the plane.
- If $k[x,y]/I$ is a local ring then $\mathrm{Spec}(k[x,y]/I)$ is non-reduced and supported at a single point (a,b).
  For example, each of $\langle x^3, y\rangle$, $\langle x^2, xy, y^2\rangle, \langle x^2+y, xy, y^2\rangle$ is an element of $\mathrm{Hilb}_3(\mathbb{A}_k^2)$ corresponding to a triple point at the origin. There are many more ideals of this sort.
  In fact, the family of all colength n ideals supported at a given point $(a,b) \in \mathbb{A}_k^2$ is a subscheme of dimension $n-1$.
- In general, $k[x,y]/I \cong k[x,y]/I_1 \times \cdots \times k[x,y]/I_r$ where each $k[x,y]/I_j$ is a local ring and the (vector space) dimensions of $k[x,y]/I_1, \ldots, k[x,y]/I_r$ sum to n.

Properties of $\mathrm{Hilb}_n(\mathbb{A}_k^2)$

Theorem: (Fogarty) $\mathrm{Hilb}_n(\mathbb{A}_k^2)$ is a non-singular, connected, integral scheme of dimension $2n$.
(The scheme structure is obtained by realizing $\mathrm{Hilb}_n(\mathbb{A}_k^2)$ as a locally closed subscheme of a Grassmannian.)

In other words, (unlike arbitrary Hilbert schemes of points) $\mathrm{Hilb}_n(\mathbb{A}_k^2)$ is very nice!

The torus $T^2 = (k^*)^2$ acts on $\mathrm{Hilb}_n(\mathbb{A}_k^2)$ by scaling x and y. That is, if $I \in \mathrm{Hilb}_n(\mathbb{A}_k^2)$ and $I = \langle f_1(x,y), \dots, f_d(x,y) \rangle$ then $(t_1, t_2) \cdot \langle f_1(x,y), \dots, f_d(x,y) \rangle = \langle f_1(t_1 x, t_2 y), \dots, f_d(t_1 x, t_2 y) \rangle$. The colength n monomial ideals are the fixed points of this action.

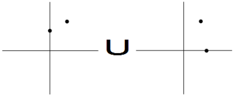
In this talk, we will be concerned with stratifying $\mathrm{Hilb}_n(\mathbb{A}_k^2)$ in a particular way. Doing so will yield finitely many (locally closed) strata which are, for example, reduced, regular in codimension 1 and stable under the T^2 action. We begin with a small example.

A stratification of $\mathrm{Hilb}_2(\mathbb{A}^2_k)$

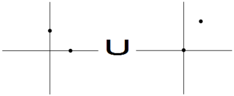
Consider $\mathrm{Hilb}_2(\mathbb{A}^2_k)$ and the reduced, T^2-invariant divisor D.

$$D = \text{“at least one point is on a coordinate axis”}$$

The two components of D will be the codimension 1 subvarieties in our stratification.

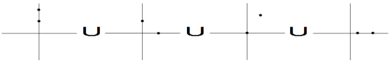


We can intersect the irreducible components of this divisor and decompose the intersection to obtain some new subschemes. These subschemes are reduced!

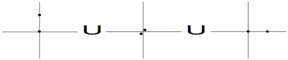


$\mathrm{Hilb}_2(\mathbb{A}^2_k)$ continued

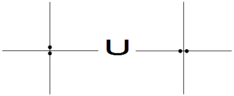
Neither irreducible component of D is regular in codimension 1. So, we include the non-R1 loci in the union of codimension 2 subvarieties to appear in the stratification of $\mathrm{Hilb}_2(\mathbb{A}^2_k)$.



Intersecting each one of these (R1) subvarieties with the union of the others and then decomposing each intersection yields the following codimension 3 subvarieties:

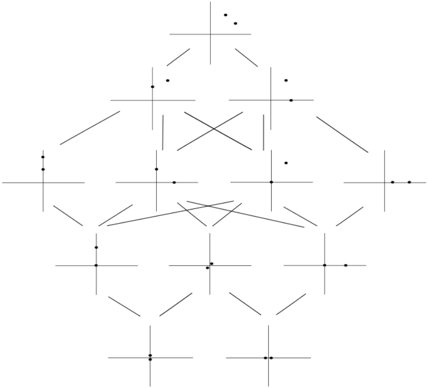


Repeating this procedure once more produces the T^2-fixed points.



$\mathrm{Hilb}_2(\mathbb{A}^2_k)$ continued

This sequence of intersecting, decomposing and including non-R1 loci produces the following stratification:



All of the subschemes appearing above turn out to be reduced – indeed, this collection of subvarieties is precisely the set of "compatibly Frobenius split" subvarieties of $\mathrm{Hilb}_2(\mathbb{A}^2_k)$.

A question

More generally, $\mathrm{Hilb}_n(\mathbb{A}^2_k)$ is Frobenius split, compatibly with the divisor, $D =$ "at least one point is on the x-axis or at least one point is on the y-axis". This allows us to ask:

What are all of the compatibly split subvarieties?

To study this question, we'll use an algorithm of Allen Knutson, Thomas Lam and David Speyer. (This algorithm makes precise the "intersect, decompose, include the non-R1 locus" procedure used above.)

Before proceeding to describe the algorithm and to investigate the above question, we should discuss some basic notions of Frobenius splitting.

Some Frobenius splitting basics

Definition: Let R be a (commutative) k-algebra and let $X = \mathrm{Spec}(R)$. Say that R (or X) is *Frobenius split* by $\phi : R \to R$ if:

$$\phi(a+b) = \phi(a) + \phi(b), \quad \phi(a^p b) = a\phi(b), \quad \phi(1) = 1$$

for any $a, b \in R$.
(Notice that ϕ is an R-module map which "splits" the Frobenius endomorphism $F : R \to R, r \mapsto r^p$. That is, $\phi \circ F = \mathrm{Id}$.)

It immediately follows from the definition that if R is Frobenius split then R has no nilpotents. So, $X = \mathrm{Spec}(R)$ is reduced.

Definition: Let $I \subset R$ be an ideal. We say that I (or $V(I)$) is *compatibly Frobenius split* if $\phi(I) \subset I$.
In this case, there is an induced splitting, $\overline{\phi} : R/I \to R/I$ and we get that I is a radical ideal.

The following are some consequences which we have already used:

- Intersections, unions and components of compatibly split subschemes are compatibly split.
- The non-$R1$ locus of any compatibly split subvariety is compatibly split.

Frobenius splittings and divisors

The definitions and results on the previous slide can be generalized to an arbitrary scheme $(X, \mathcal{O}_X)$. (See Brion-Kumar.) We won't discuss this. However, it is necessary for us to consider the following situation:

Let X be a non-singular (or normal) scheme.

Recall that an *anticanonical divisor* of X is the divisor associated to a section of $\bigwedge^{\dim(X)}(TX)$.

These divisors are important to us for the following reason:

Fact: Certain anticanonical divisors D induce (in a specific way) a Frobenius splitting on X such that the compatibly split codimension 1 subvarieties are the components of the divisor's support.

In fact, *all* compatibly split subvarieties are contained inside of D or inside of the singular locus of $X \setminus D$. (Kumar-Mehta)

Some divisors that induce splittings

Theorem: (Lakshmibai-Mehta-Parameswaran)
Let $f \in k[x_1, \ldots, x_n]$. If there is a term order on $k[x_1, \ldots, x_n]$ such that $\mathrm{init}(f) = x_1 x_2 \cdots x_n$ then $V(f)$ induces a splitting on $\mathbb{A}^n$ that compatibly splits $V(f)$.

Example: Affine space is Frobenius split compatibly with $V(x_1 x_2 \ldots x_n)$. This splitting of $\mathbb{A}^n$ is called the *standard splitting*. By decomposing the components of the divisor, intersecting the pieces, decomposing the intersections, etc., we see that the collection of coordinate subspaces is precisely the set of compatibly split subvarieties.

Theorem: (Kumar-Thomsen) The anticanonical divisor described by "at least one point is on an axis" induces a Frobenius splitting on $\mathrm{Hilb}_n(\mathbb{A}^2_k)$.

We are now ready to describe the Knutson-Lam-Speyer algorithm.

An algorithm

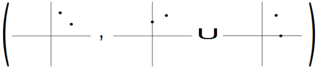
Algorithm(Knutson-Lam-Speyer)
Input: $(X, \partial X)$ where X is Frobenius split and ∂X is the anticanonical divisor which induces the splitting.
Output: Suppose that $\partial X = D_1 \cup \cdots \cup D_r$. Let $E_i = D_1 \cup \cdots \cup \hat{D}_i \cup \cdots \cup D_n$. There are two cases.

1. If X is regular in codimension 1, then return $(D_1, D_1 \cap E_1)$,. . . ,$(D_n, D_n \cap E_n)$.
2. If X is not $R1$, return $(\tilde{X}, \nu^{-1}(\partial X \cup X_{\mathrm{non}\text{-}R1}))$ where $\nu : \tilde{X} \to X$ is the normalization of X.

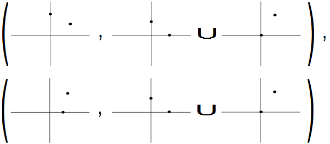
Repeat until neither 1. nor 2. can be applied. When finished, map all subvarieties back to the original Frobenius split variety to obtain a list of many (for large p) compatibly split subvarieties.
At each stage of the algorithm, check if $\exists$ a component of the singular locus that is both compatibly split and of codimension ≥ 2. (Hard!)
If so, add it (and its compatibly split subvarieties) to the list. The final list consists of all compatibly split subvarieties of $(X, \partial X)$.

Back to $\mathrm{Hilb}_2(\mathbb{A}^2_k)$

As an example of the algorithm, we consider (again) the case of $\mathrm{Hilb}_2(\mathbb{A}^2_k)$.
Start with $(\mathrm{Hilb}_2(\mathbb{A}^2_k), D)$ where D is as before.

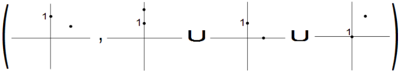


Apply 1.



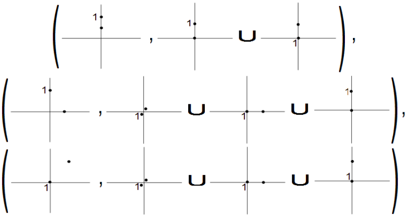
Due to the symmetry, continue with just the first of the two pairs.

Next, recall that the components of D are not regular in codimension 1. Apply 2.



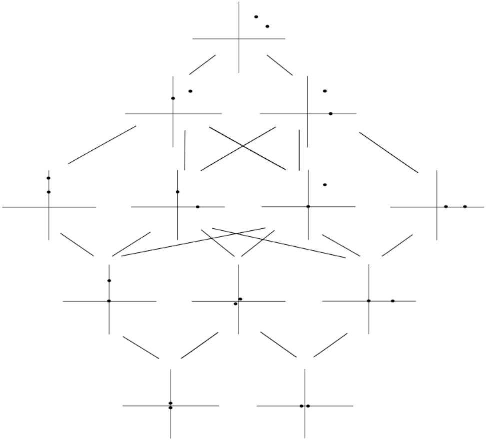
Continued

Apply 1.



Applying 1. once more obtains the preimage of the T^2-fixed points under the normalization map $\nu : X_n \to \mathrm{Hilb}_2(\mathbb{A}_k^2)$ where X_n denotes the isospectral Hilbert scheme (i.e. the scheme of *labelled* points in the affine plane).

The compatibly split subvarieties of $\mathrm{Hilb}_2(\mathbb{A}^2_k)$

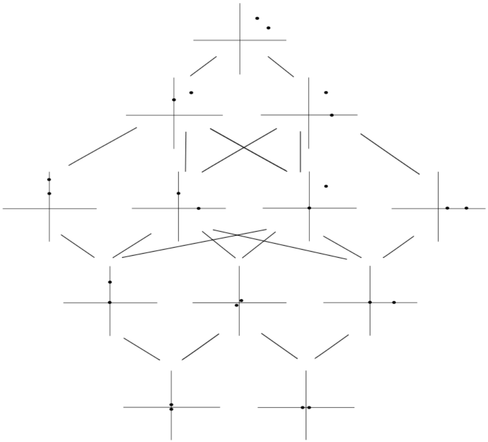


A remark

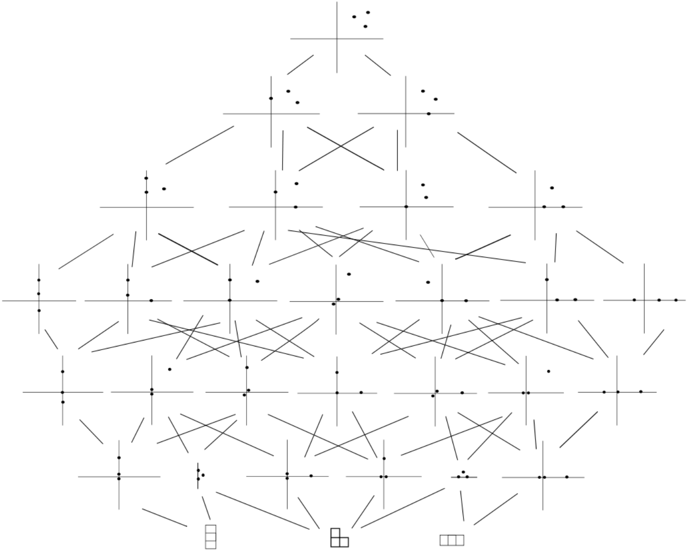
In the case of $\mathrm{Hilb}_2(\mathbb{A}_k^2)$, repeatedly applying steps 1. and 2. was enough to find all compatibly split subvarieties. This is not always the case; it is necessary to check for compatibly split subvarieties inside of the singular locus at each stage of the algorithm.

Example: Let $p \equiv 1 \pmod 3$ and let $\{x^3 + y^3 + z^3 = 0\}$ be the divisor that determines the splitting of $\mathbb{A}_k^3$. In this case, $(X, \partial X) = (\mathbb{A}_k^3, \{x^3 + y^3 + z^3 = 0\})$. Neither 1. nor 2. can be applied. However, the origin is compatibly split.

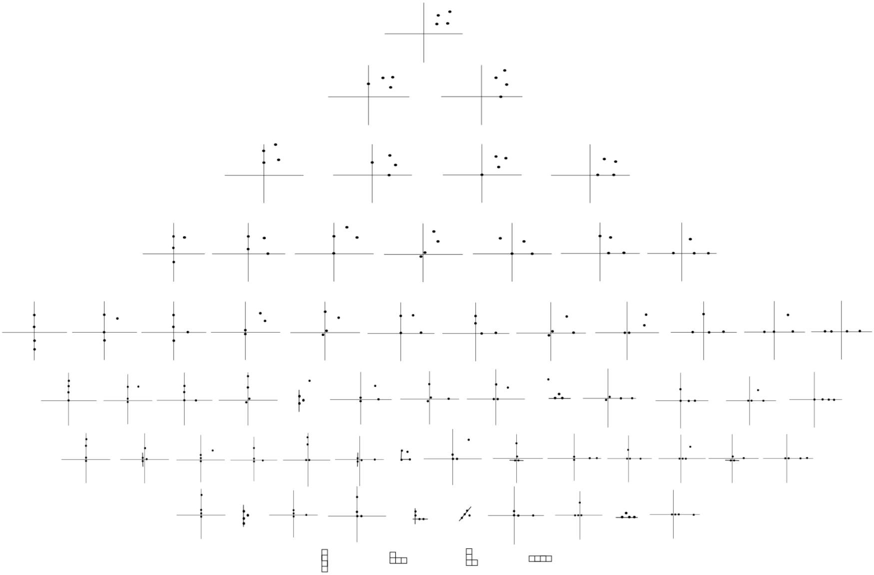
The compatibly split subvarieties of $\mathrm{Hilb}_2(\mathbb{A}_k^2)$



The compatibly split subvarieties of $\mathrm{Hilb}_3(\mathbb{A}^2_k)$



The compatibly split subvarieties of $\mathrm{Hilb}_4(\mathbb{A}^2_k)$



More generally

- Notice that the poset of compatibly split subvarieties for $\mathrm{Hilb}_{n_1}(\mathbb{A}^2_k)$ appears inside the poset for $\mathrm{Hilb}_{n_2}(\mathbb{A}^2_k)$ $(n_1 < n_2)$ by adding some more points.
- Also, we can see from the algorithm that the subvariety of $\mathrm{Hilb}_n(\mathbb{A}^2_k)$ corresponding to "q points on the x-axis and r points on the y-axis" for $q + r \leq n$ is compatibly split.
- It is difficult to say much more about the general case. It is even hard to guess which monomial ideals are compatibly split for arbitrary n. Recall that for each of $\mathrm{Hilb}_2(\mathbb{A}^2_k)$ and $\mathrm{Hilb}_3(\mathbb{A}^2_k)$, all of the monomial ideals are compatibly split. However, this doesn't happen in $\mathrm{Hilb}_4(\mathbb{A}^2_k)$; that is, the variety consisting of the point $\langle x^2, y^2 \rangle$ is not split.

For the rest of the talk, we restrict to an open affine patch of the Hilbert scheme and see that things are more understandable.

An open affine patch of $\mathrm{Hilb}_n(\mathbb{A}^2_k)$

Let λ be a monomial ideal in $\mathrm{Hilb}_n(\mathbb{A}^2_k)$. Let U_λ denote the set of all $I \in \mathrm{Hilb}_n(\mathbb{A}^2_k)$ such that the monomials outside λ form a vector space basis of $k[x,y]/I$.

Example: Let $\lambda = \langle y^2, x\rangle \in \mathrm{Hilb}_2(\mathbb{A}^2_k)$. Then $I = \langle y^2 + y, x + 2\rangle \in U_\lambda$ as $\{1, y\}$ spans the vector space $k[x,y]/I$.

Set $\lambda = \langle y^n, x\rangle$ for the remainder of the talk.

Fix the Lex term order with $x \gg y$. The elements of U_λ are ideals generated by polynomials of the form

$$\begin{array}{lllllll}
y^n & - \; b_1 y^{n-1} & - \; b_2 y^{n-2} & - \; \cdots & - \; b_{n-1} y & - \; b_n \\
xy^{n-1} & - \; a_1 y^{n-1} & - \; c_{12} y^{n-2} & - \; \cdots & - \; c_{1(n-1)} y & - \; c_{1n} \\
xy^{n-2} & - \; a_2 y^{n-1} & - \; c_{22} y^{n-2} & - \; \cdots & - \; c_{2(n-1)} y & - \; c_{2n} \\
 & & \vdots & & & \\
xy & - \; a_{n-1} y^{n-1} & - \; c_{(n-1)2} y^{n-2} & - \; \cdots & - \; c_{(n-1)(n-1)} y & - \; c_{(n-1)n} \\
x & - \; a_n y^{n-1} & - \; c_{n2} y^{n-2} & - \; \cdots & - \; c_{n(n-1)} y & - \; c_{nn}
\end{array}$$

where each c_{ij} is a polynomial in $a_1, \ldots, a_n, b_1, \ldots, b_n$.

Thus, $U_\lambda \cong \mathbb{A}^{2n} = \mathrm{Spec}(k[a_1, b_1, \ldots, a_n, b_n])$.

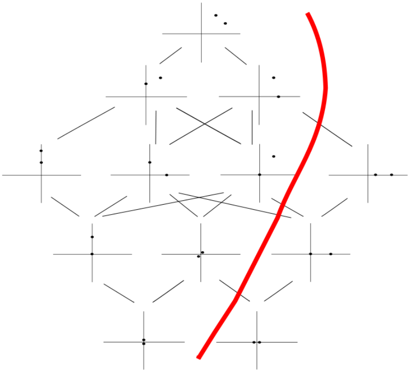
The splitting of U_λ

Let X be a (normal) scheme that is Frobenius split compatibly with an anticanonical divisor D. Let V be an open subscheme of X. Then, $D \cap V$ induces a splitting of V and the compatibly split subvarieties of V are the $Y \cap V$ where Y is a compatibly split subvariety of X.

In our situation, this says that there is a Frobenius splitting of U_λ induced by the Frobenius splitting of $\mathrm{Hilb}_n(\mathbb{A}^2_k)$. We may therefore ask the (hopefully easier) question,

"What are all of the compatibly split subvarieties of U_λ with this induced splitting?"

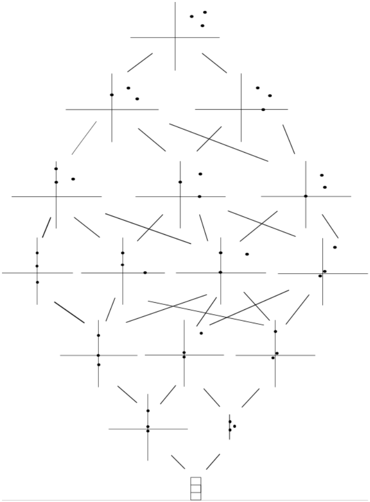
Compatibly split subschemes of $U_\lambda \subset \mathrm{Hilb}_2(\mathbb{A}^2_k)$



The subvarieties to the left of the red curve have non-trivial intersection with $U_\lambda \subset \mathrm{Hilb}_2(\mathbb{A}^2_k)$.

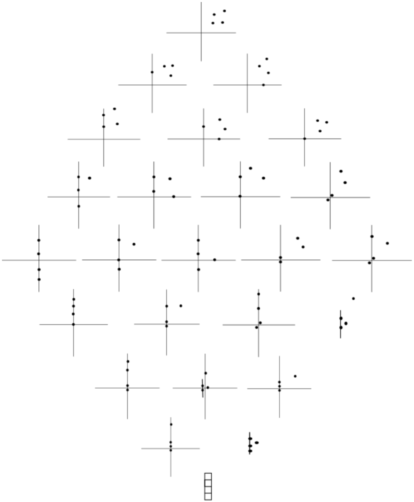
Notice that the poset of compatibly split subvarieties of U_λ is a square.

Compatibly split subschemes of $U_\lambda \subset \mathrm{Hilb}_3(\mathbb{A}^2_k)$



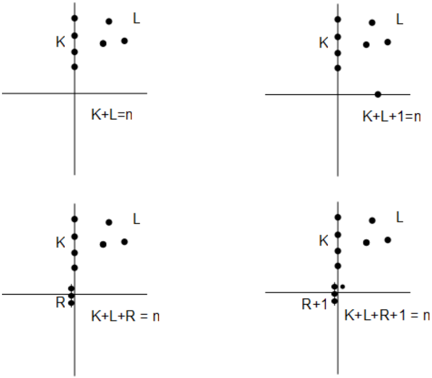
Again, we get a square.

Compatibly split subschemes of $U_\lambda \subset \mathrm{Hilb}_4(\mathbb{A}^2_k)$



Compatibly split subvarieties of $U_\lambda \subset \mathrm{Hilb}_n(\mathbb{A}^2_k)$

More generally, we get (at least) the following types of compatibly split subvarieties:



It can be checked, by counting dimensions, that the stratification of U_λ by these subvarieties again has the shape of a square.

To better understand the compatibly split subvarieties of U_λ, it is useful to understand the anticanonical that determines the splitting. We do this next.

The divisor that determines the splitting of $U_\lambda \cong \mathbb{A}_k^{2n}$

Recall that $\mathrm{Hilb}_n(\mathbb{A}_k^2)$ is Frobenius split compatibly with the divisor $D =$"at least one point is on an axis". This has two components and so, $D \cap U_\lambda$ should be defined by the product of two irreducible polynomials.

Example: Consider $U_\lambda \subset \mathrm{Hilb}_2(\mathbb{A}_k^2)$. Then, every ideal in U_λ can be written as:

$$I = \left\langle \begin{array}{ccccc} y^2 & - & b_1 y & - & b_2 \\ xy & - & a_1 y & - & a_2 b_2 \\ x & - & a_2 y & - & (a_1 - b_1 a_2) \end{array} \right\rangle$$

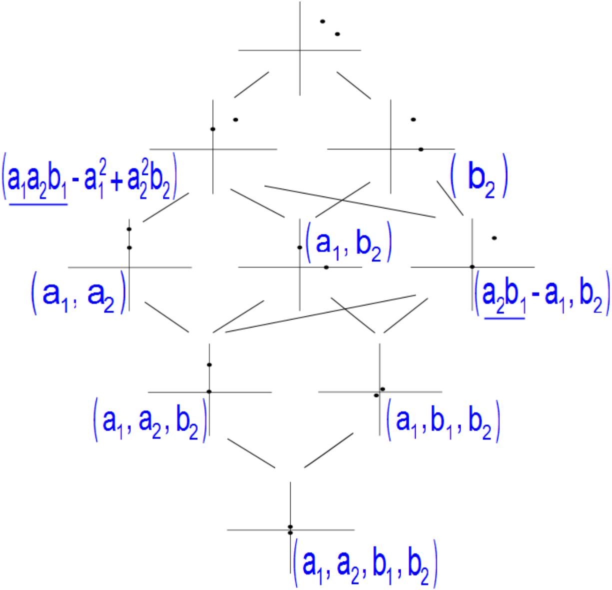
If there is at least one point on the x-axis, then $I + \langle y \rangle \neq \langle 1 \rangle$. Thus, $-b_2 = 0$. Similarly, if there is a point on the y-axis then $I + \langle x \rangle \neq \langle 1 \rangle$ and

$$\begin{vmatrix} -a_1 & -a_2 b_2 \\ -a_2 & -(a_1 - b_1 a_2) \end{vmatrix} = 0$$

So, $V(a_1 b_1 a_2 b_2 - a_1^2 b_2 + a_2^2 b_2^2)$ determines the splitting of U_λ.

Compatibly split subvarieties of $U_\lambda \subset \mathrm{Hilb}_2(\mathbb{A}^2_k)$

We may apply the Knutson-Lam-Speyer algorithm to obtain:

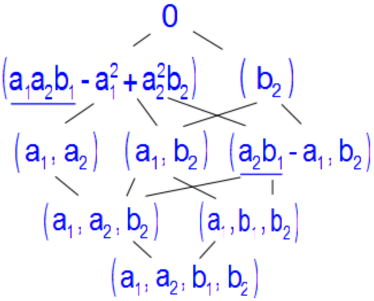


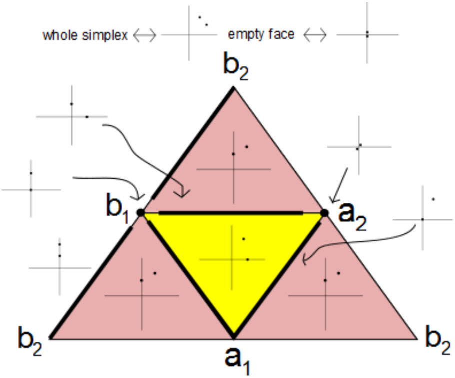
Initial ideals and simplicial complexes

With respect to the term order

$$\mathrm{revlex}_{b_2},\ \mathrm{lex}_{a_2},\ \mathrm{revlex}_{b_1},\ \mathrm{lex}_{a_1},$$

$\mathrm{init}(I)$, for I compatibly split, is a squarefree monomial ideal. We can therefore associate a simplicial complex to each $\mathrm{init}(I)$.



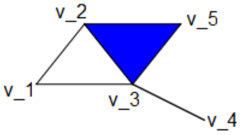


Note that the picture on the right is a 3-simplex that has been "unfolded" to better see all of the faces.

Squarefree monomial ideals

There is a one-to-one correspondence between squarefree monomial ideals and simplicial complexes.

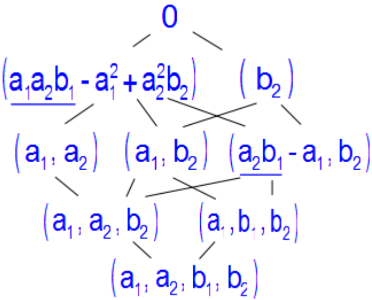
Example:
Let Δ be the following simplicial complex on the vertex set $\{v_1, \ldots, v_6\}$.

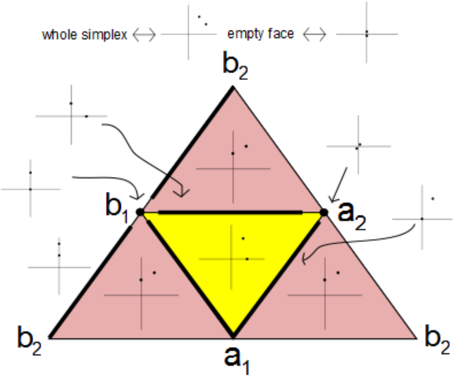


The squarefree monomial ideal (or "Stanley-Reisner" ideal) of Δ is the ideal I_Δ generated by the minimal non-faces of Δ.
In the example, $I_\Delta = \langle v_6, v_1v_4, v_1v_5, v_2v_4, v_4v_5, v_1v_2v_3 \rangle$.

On the other hand, given I_Δ, we can decompose to obtain $\langle v_1, v_2, v_5, v_6 \rangle \cap \langle v_3, v_4, v_5, v_6 \rangle \cap \langle v_2, v_4, v_5, v_6 \rangle \cap \langle v_1, v_4, v_6 \rangle$. The (maximal dimensional) faces of Δ are then $v_3v_4, v_1v_2, v_1v_3, v_2v_3v_5$.

Back to our example





Notice that all compatibly split subvarieties degenerate to the Stanley-Reisner scheme of a ball. This is interesting to note; we'll come back to this later.

More generally

Theorem: (Knutson, building on LMP) Fix a term order on $k[x_1, \ldots, x_n]$. Let $f \in k[x_1, \ldots, x_n]$ be a degree n polynomial such that $\mathrm{init}(f) = \prod_i x_i$. (Thus f induces a splitting on $\mathbb{A}^n_k$.)
If I is compatibly split with respect to this splitting, then $\mathrm{init}(I)$ is compatibly split with respect to the standard splitting.
Thus, for each compatibly split ideal I, $\mathrm{init}(I)$ is a squarefree monomial ideal.

This theorem applies in our situation.

A polynomial and a term order

More generally, recall that elements of U_λ are ideals generated by polynomials of the form

$$\begin{array}{ccccccccccc}
y^n & - & b_1 y^{n-1} & - & b_2 y^{n-2} & - & \cdots & - & b_{n-1} y & - & b_n \\
xy^{n-1} & - & a_1 y^{n-1} & - & c_{12} y^{n-2} & - & \cdots & - & c_{1(n-1)} y & - & c_{1n} \\
xy^{n-2} & - & a_2 y^{n-1} & - & c_{22} y^{n-2} & - & \cdots & - & c_{2(n-1)} y & - & c_{2n} \\
 & & & & \vdots & & & & & & \\
xy & - & a_{n-1} y^{n-1} & - & c_{(n-1)2} y^{n-2} & - & \cdots & - & c_{(n-1)(n-1)} y & - & c_{(n-1)n} \\
x & - & a_n y^{n-1} & - & c_{n2} y^{n-2} & - & \cdots & - & c_{n(n-1)} y & - & c_{nn}
\end{array}$$

where each c_{ij} is a polynomial in $a_1, \dots, a_n, b_1, \dots, b_n$.
Let M_n be the matrix of coefficients $(-c_{ij})_{1 \le i,j \le n}$ where $c_{i1} = a_i$.

Then, the divisor that determines the splitting on U_λ is given by $V(f_n)$ where $f_n = -b_n(\det M_n)$. Furthermore, the term order

$$\mathrm{revlex}_{b_n},\ \mathrm{lex}_{a_n}, \dots,\ \mathrm{revlex}_{b_1},\ \mathrm{lex}_{a_1},$$

is such that $\mathrm{init}(f_n) = b_n a_n \cdots b_1 a_1$.

An example

We have:

$$M_2 = \begin{pmatrix} -a_1 & -a_2 b_2 \\ -a_2 & -(a_1 - b_1 a_2) \end{pmatrix}, \quad M_3 = \begin{pmatrix} -a_1 & -(a_2 b_2 + a_3 b_3) & -a_2 b_3 \\ -a_2 & -(a_1 - b_1 a_2) & -a_3 b_3 \\ -a_3 & -(a_2 - b_1 a_3) & -(a_1 - b_1 a_2 - b_2 a_3) \end{pmatrix}$$

$$M_4 = \begin{pmatrix} -a_1 & -(a_2 b_2 + a_3 b_3 + a_4 b_4) & -(a_2 b_3 + a_3 b_4) & -a_2 b_4 \\ -a_2 & -(a_1 - b_1 a_2) & -(a_3 b_3 + a_4 b_4) & -a_3 b_4 \\ -a_3 & -(a_2 - b_1 a_3) & -(a_1 - b_1 a_2 - b_2 a_3) & -a_4 b_4 \\ -a_4 & -(a_3 - b_1 a_4) & -(a_2 - b_1 a_3 - b_2 a_4) & -(a_1 - b_1 a_2 - b_2 a_3 - b_3 a_4) \end{pmatrix}$$

Computing the determinant of M_4 using cofactors along the last column, we get:

$$\det M_4 = (M_4)_{44}(\det M_3) + b_4(\cdots).$$

Taking the terms with the smallest power of b_4 (i.e. computing $\mathrm{init}_{\mathrm{revlex}_{b_4}}(\det M_4)$) yields:

$$(M_4)_{44}(\det M_3).$$

Taking $\mathrm{init}_{\mathrm{lex}_{a_4}}$ of this polynomial yields:

$$a_4 b_3(\det M_3).$$

An example (continued)

We have: $\mathrm{init}_{\mathrm{lex}_{a_4}} \mathrm{init}_{\mathrm{revlex}_{b_4}} (\det\ M_4) = a_4 b_3 (\det\ M_3)$.
Recall also that:

$$M_2 = \begin{pmatrix} -a_1 & -a_2 b_2 \\ -a_2 & -(a_1 - b_1 a_2) \end{pmatrix}, \quad M_3 = \begin{pmatrix} -a_1 & -(a_2 b_2 + a_3 b_3) & -a_2 b_3 \\ -a_2 & -(a_1 - b_1 a_2) & -a_3 b_3 \\ -a_3 & -(a_2 - b_1 a_3) & -(a_1 - b_1 a_2 - b_2 a_3) \end{pmatrix}.$$

Taking $\mathrm{init}_{\mathrm{revlex}_{b_3}}$ and then $\mathrm{init}_{\mathrm{lex}_{a_3}}$ of $a_4 b_3 (\det\ M_3)$ produces:

$$a_4 b_3 a_3 b_2 (\det\ M_2)$$

Doing this once more with with b_2 and a_2 yields:

$$a_4 b_3 a_3 b_2 a_2 b_1 (-a_1)$$

Thus, under the term order

$$\mathrm{revlex}_{b_4},\ \mathrm{lex}_{a_4},\ \mathrm{revlex}_{b_3},\ \mathrm{lex}_{a_3},\ \mathrm{revlex}_{b_2},\ \mathrm{lex}_{a_2},\ \mathrm{revlex}_{b_1},\ \mathrm{lex}_{a_1},$$

we get that $\mathrm{init}(f_4) = \mathrm{init}(-b_4(\det\ M_4)) = b_4 a_4 b_3 a_3 b_2 a_2 b_1 a_1$.

Associating simplicial complexes to subvarieties

Because we have a term order such that $\mathrm{init}(f_n) = b_n a_n \cdots b_1 a_1$, Knutson's theorem tell us that we can associate a simplicial complex to each compatibly split subvariety.

We've seen that we can do this as follows:

Take the ideal defining the subvariety, compute a Gröbner basis with respect to our term order, read off the initial ideal and then associate a simplicial complex.

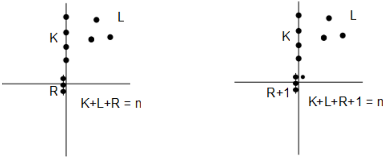
Doing this each time would be somewhat difficult/annoying. Luckily, this work isn't necessary. Rather, we can do everything combinatorially.

Associating "words" to subvarieties

To each compatibly split subvariety we associate "words" in the following "letters":

$$(1) \quad ab, \quad (2) \quad \hat{a}b, \quad (3) \quad aab, \quad (4) \quad aa, \quad (5) \quad \hat{a}$$

Let X be a compatibly split subvariety of $U_\lambda \subset \mathrm{Hilb}_n(\mathbb{A}^2_k)$. Suppose that X has L free points, K points freely on the y-axis and R points "vertically stacked" at the origin. For example:



Then, X is associated to the collection of words of the form

$$(\text{word in } (1),\ (2),\ (3)) \ \mid \ (\text{word in } (4),\ (5)) \ \mid \ (a \text{ iff } \text{"}R+1\text{" at origin})$$

such that

$$\#(1) + \#(3) + \#(4) = L, \quad \#(2) + \#(3) = K, \quad \#(4) + \#(5) = R.$$

Point: These words are in one-to-one correspondence with the facets of the simpicial complex associated to X.

Note: I am missing some details in the proof of the above assertion.

Combinatorial (and thus geometric) properties

Using the above recipe to associate a simplicial complex to a compatibly split subvariety $X \subset U_\lambda$, we obtain the following:

Suppose that X has no free points:
In this case, $\mathrm{init}(X)$ is the Stanley-Reisner scheme of a simplex. Thus, $\mathrm{init}(X)$ is affine space. By semicontinuity, X is non-singular.

Suppose that X has at least one free point:

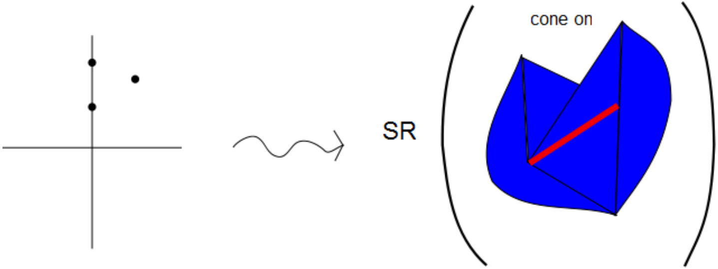
1. If X has no points freely on the y-axis, then $\mathrm{init}(X)$ is the Stanley-Reisner scheme of a (vertex decomposable) ball. And, $\partial(X)$, the union of codimension 1 compatibly split subvarieties of X, degenerates onto the boundary sphere.
2. If X has exactly one point freely on the y-axis, then $\mathrm{init}(X)$ is the Stanley-Reisner scheme of a (vertex decomposable) ball.

In each of the above two cases, we get that $\mathrm{init}(X)$ is Cohen-Macaulay. Therefore, so is X.

The remaining case

If X has $K \geq 2$ points freely on the y axis then things aren't as nice. In particular, the non-$R1$ locus of X degenerates to a simplicial complex that is contained in precisely $K+1$ facets of the complex associated to X.

Example:



Note that the cone on the red edge is the simplicial complex associated to the subvariety "3 points are on the y-axis". This is the non-$R1$ locus of X.

Thank You.