Computing the infinite

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Tohoku University & Ghent University

LICS11, June 22, 2011, Fields Institute, Toronto

[1]This research is generously supported by the John Templeton Foundation.

Goal & Motivation

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Motivation: Erret Bishop (and others) have derided NSA for its 'lack of computational content'.

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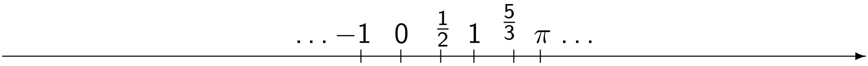

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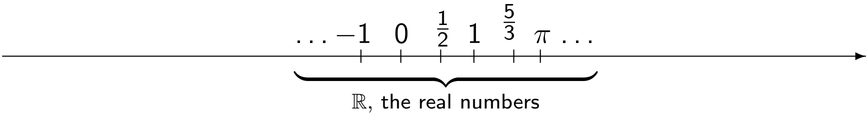

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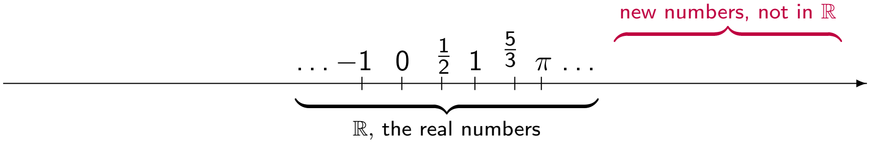

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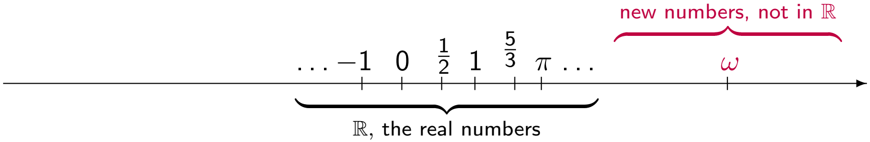

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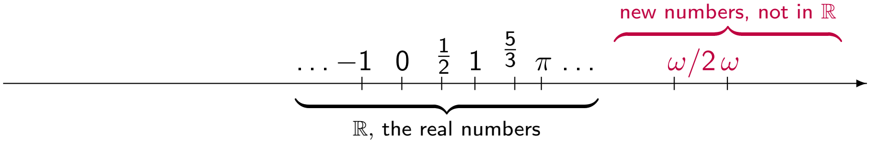

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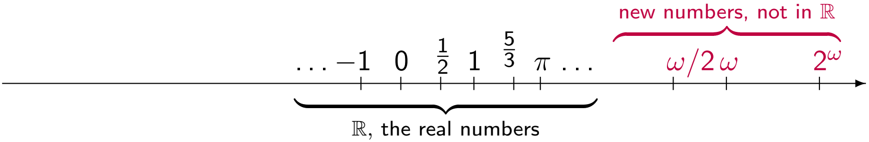

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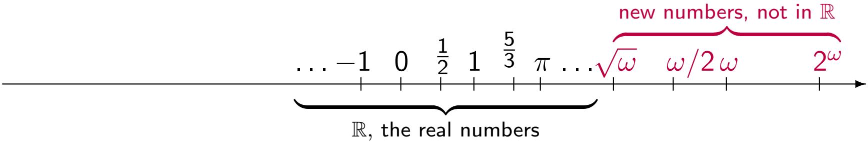

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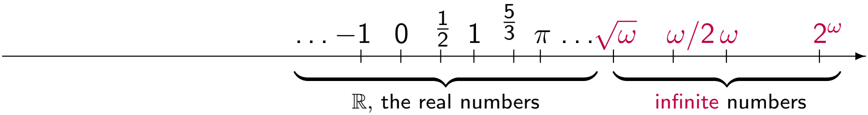

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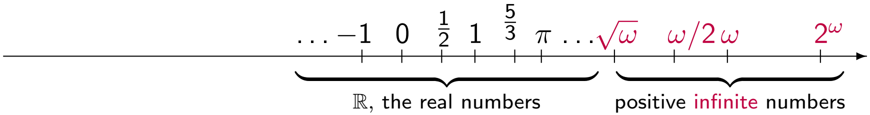

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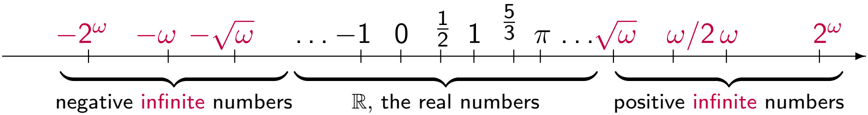

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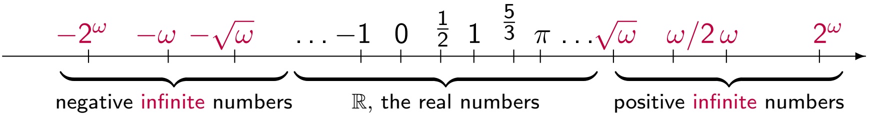

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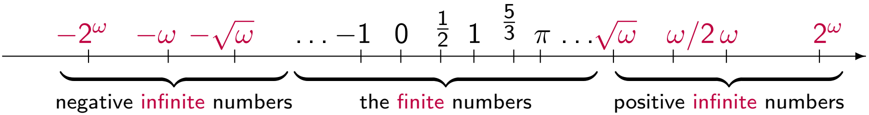

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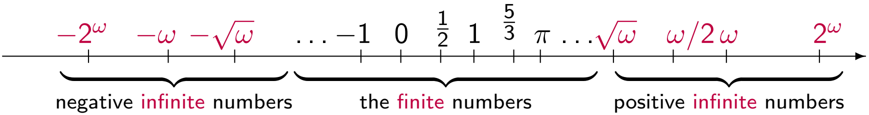

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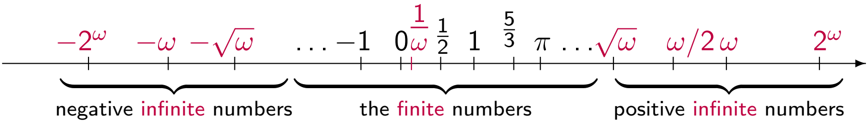

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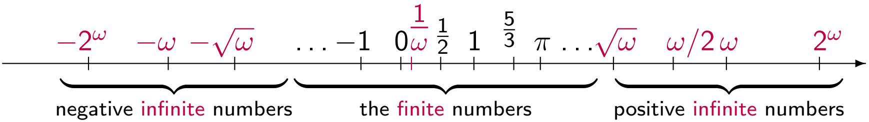

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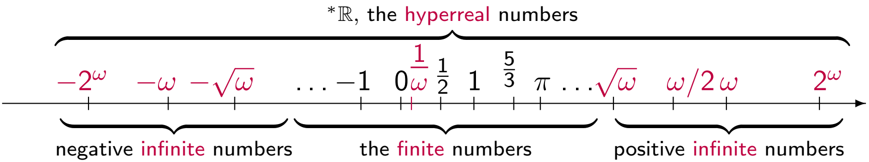

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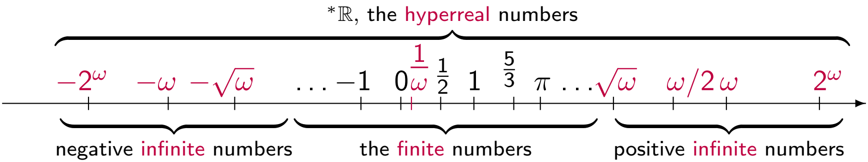

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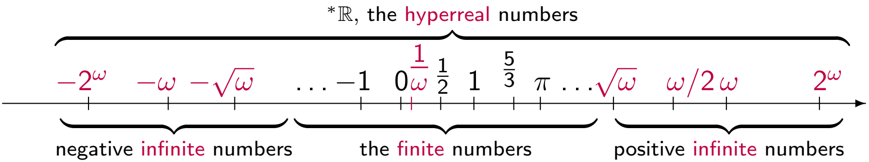

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The set $\mathbb{N}$ is extended to ${}^*\mathbb{N} = \{\underbrace{0, 1, 2, 3, \dots}_{\mathbb{N}}, \dots, \omega - 1, \omega, \omega + 1, \dots\}$

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Theorem

The Δ_1-sets (=Turing computable) are exactly the ω-invariant sets.

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Generalizes to any finite Turing degree. Comes from RM.

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Compare $\mathbb{N}$ and $\mathcal{N}$.

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