Concurrent Strategies

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The next-generation domain theory? An intensional theory to capture the ways *of computing, to near operational concerns and reasoning. An event-based theory.*

A new result characterizing (nondeterministic) concurrent strategies

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Representations of traditional domains

What is the information order? What are the ‘units’ of information?
Two answers:

(‘Topological’) [Scott]: *Propositions* about finite properties;
more information corresponds to more propositions being true.
Functions are ordered pointwise. Can represent domains via logical theories (‘Information systems’, ‘Logic of domains’).

(‘Temporal’) [Berry]: *Events* (atomic actions);
more information corresponds to more events having occurred.
Intensional ‘stable order’ on ‘stable’ functions. (‘Stable domain theory’)
Can represent Berry’s dI domains as event structures.

Event structures

An **event structure** comprises $(E, \mathrm{Con}, \leq)$, consisting of a set of *events* E

- partially ordered by $\leq$, the **causal dependency relation**, and

- a nonempty family Con of finite subsets of E, the **consistency relation**,

which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$

$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$

$$Y \subseteq X \in \mathrm{Con} \Rightarrow Y \in \mathrm{Con}, \quad \text{and}$$

$$X \in \mathrm{Con} \;\&\; e \leq e' \in X \Rightarrow X \cup \{e\} \in \mathrm{Con}.$$

Say e, e' are **concurrent** if $\{e, e'\} \in \mathrm{Con} \;\&\; e \not\leq e' \;\&\; e' \not\leq e$.
In games the relation of **immediate dependency** $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, will play an important role.

Configurations of an event structure

The **configurations**, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq_{\text{fin}} x.\ X \in \text{Con}$ and

Down-closed: $\forall e, e'.\ e' \leq e \in x \Rightarrow e' \in x$.

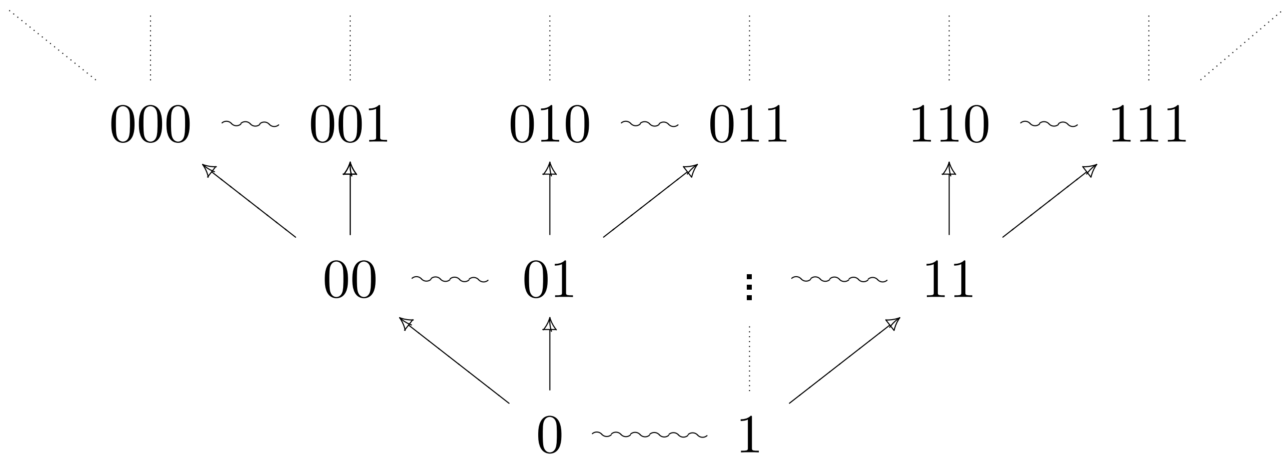
For an event e the set $[e] =_{\text{def}} \{e' \in E \mid e' \leq e\}$ is a configuration describing the whole causal history of the event e.

$x \subseteq x'$, *i.e.* x is a sub-configuration of x', means that x is a sub-history of x'.

If E is countable, $(\mathcal{C}^\infty(E), \subseteq)$ is a dI domain (and all such are so obtained).

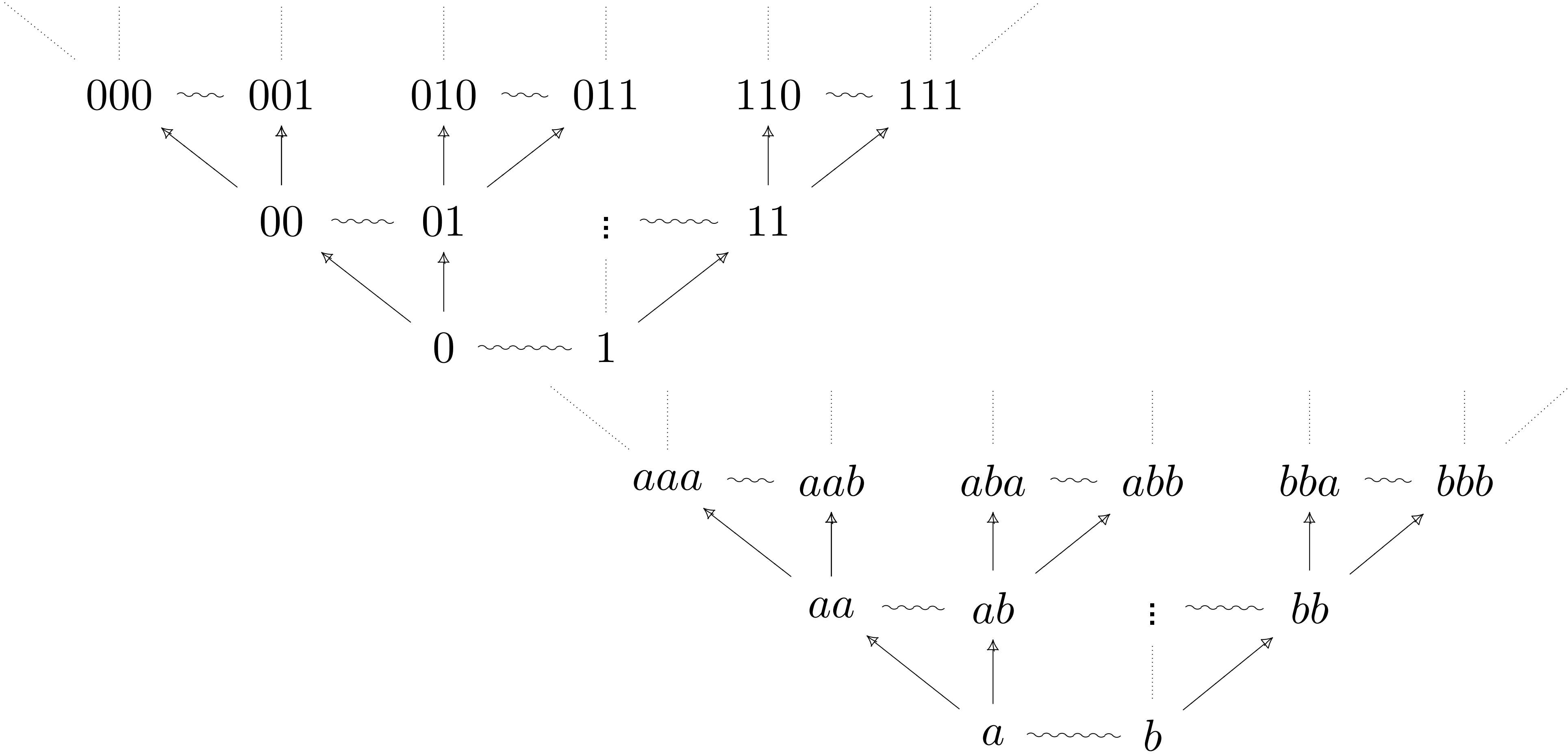
Here concentrate on the **finite configurations** $\mathcal{C}(E)$.

Event structures as types, e.g., Streams as event structures



$\rightsquigarrow$ conflict (inconsistency) $\qquad$ $\longrightarrow$ immed. causal dependency

Simple parallel composition



Event structures as processes

- Semantics of synchronising processes [Hoare, Milner] can be expressed in terms of universal constructions on event structures, and other models.
- Relations between models via adjunctions.

In this context, a **simulation map** of event structures $f : E \to E'$
is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$

$fx \in \mathcal{C}(E')$ and

if $e_1, e_2 \in x$ and $f(e_1) = f(e_2)$, then $e_1 = e_2$. *('event linearity')*

Idea: *the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined.*

Process constructions on event structures

"Partial synchronous" product: $A \times B$ with projections Π_1 and Π_2, *cf.* CCS synchronized composition where all events of A *can* synchronize with all events of B. ($\mathit{Hard\ to\ construct\ directly\ so\ use\ e.g.\ stable\ families.}$)

Restriction: $E \upharpoonright R$, the restriction of an event structure E to a subset of events R, has events $E' = \{e \in E \mid [e] \subseteq R\}$ with causal dependency and consistency restricted from E.

Synchronized compositions: restrictions of products $A \times B \upharpoonright R$, where R specifies the allowed synchronized and unsynchronized events.

Projection: Let E be an event structure. Let V be a subset of 'visible' events. The *projection* of E on V, $E{\downarrow}V$, has events V with causal dependency and consistency restricted from E.

Product—an example

$$
\begin{tikzcd}
b & & & & (b,*) \arrow[rr, bend left, squiggly, no head] & (b,*) \arrow[r, squiggly, no head] & (b,c) \arrow[d, squiggly, no head] \\
a \arrow[u] & \times & c & = & (a,*) \arrow[u] \arrow[rru] \arrow[r, squiggly, no head] & (a,c) \arrow[u] \arrow[r, squiggly, no head] & (*,c)
\end{tikzcd}
$$

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Concurrent games

Basics

Games and strategies are represented by **event structures with polarity**, an event structure in which all events carry a polarity $+/-$, respected by maps.

The two polarities $+$ and $-$ express the dichotomy:
player/opponent; process/environment; ally/enemy.

Dual, $E^\perp$, of an event structure with polarity E is a copy of the event structure E with a reversal of polarities; $\overline{e} \in E^\perp$ is complement of $e \in E$, and *vice versa*.

A (nondeterministic) concurrent **pre-strategy** in game A is a total map

$$\sigma : S \to A$$

of event structures with polarity (*a nondeterministic play in game A*).

Pre-strategies as arrows

A pre-strategy $\sigma : A \nrightarrow B$ is a total map of event structures with polarity

$$\sigma : S \to A^\perp \parallel B\,.$$

It determines a span of event structures with polarity

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ A^\perp & & B \end{tikzcd}$$

where σ_1, σ_2 are *partial* maps of event structures with polarity; one and only one of σ_1, σ_2 is defined on each event of S.

Composing pre-strategies

Two pre-strategies $\sigma : A \rightarrow\!\!\!\!\!+ B$ and $\tau : B \rightarrow\!\!\!\!\!+ C$ as spans:

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & & & T \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ A^\perp & & B & B^\perp & & C\,. \end{tikzcd}$$

Their **composition**

$$\begin{tikzcd} & T \odot S \arrow[dl, "(\tau\odot\sigma)_1"'] \arrow[dr, "(\tau\odot\sigma)_2"] & \\ A^\perp & & C \end{tikzcd}$$

where $T \odot S =_{\mathrm{def}} (S \times T \upharpoonright \textbf{Syn}) \downarrow \textbf{Vis}$ where ...

$$
\begin{tikzcd}
 & & & S \times T \arrow[dll, "\Pi_1"'] \arrow[drr, "\Pi_2"] & & & \\
 & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & & & & T \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\
A^{\perp} & & B & & B^{\perp} & & C
\end{tikzcd}
$$

Their composition: $T \odot S =_{\mathrm{def}} (S \times T \upharpoonright \textbf{Syn}) \downarrow \textbf{Vis}$ where

$$
\begin{aligned}
\textbf{Syn} \;=\; & \{p \in S \times T \;\mid\; \sigma_1\Pi_1(p) \text{ is defined } \& \ \Pi_2(p) \text{ is undefined}\} \;\cup \\
& \{p \in S \times T \;\mid\; \sigma_2\Pi_1(p) = \overline{\tau_1\Pi_2(p)} \text{ with both defined}\} \;\cup \\
& \{p \in S \times T \;\mid\; \tau_2\Pi_2(p) \text{ is defined } \& \ \Pi_1(p) \text{ is undefined}\}\,, \\
\textbf{Vis} \;=\; & \{p \in S \times T \upharpoonright \textbf{Syn} \;\mid\; \sigma_1\Pi_1(p) \text{ is defined}\} \;\cup \\
& \{p \in S \times T \upharpoonright \textbf{Syn} \;\mid\; \tau_2\Pi_2(p) \text{ is defined}\}\,.
\end{aligned}
$$

Concurrent copy-cat

Identities on games A are given by **copy-cat strategies** $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \parallel A$ —strategies for player based on copying the latest moves made by opponent.

$\mathbb{C}\mathbb{C}_A$ comprises $A^\perp \parallel A$ with *extra* causal dependency

$$\overline{c} \leq_{\mathbb{C}\mathbb{C}_A} c \quad \text{if} \quad pol_{A^\perp \parallel A}(c) = +$$

where $\overline{c} \leftrightarrow c$ is the correspondence between events in $A^\perp$ and A.
Map $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$ acts as identity on the underlying sets of events.

Then,

$$x \in \mathcal{C}(\mathbb{C}\mathbb{C}_A) \quad \text{iff} \quad x \in \mathcal{C}(A^\perp \parallel A) \ \& \ \forall c \in x.\ pol_{A^\perp \parallel A}(c) = + \ \Rightarrow \ \overline{c} \in x\,.$$

Copy-cat—an example

$$\mathbb{C}\mathbb{C}_A$$

$$\begin{tikzcd}
A^{\perp} & & & A \\
\overline{a}_2 & \ominus \arrow[r, dashed] & \oplus & a_2 \\
\overline{a}_1 & \oplus \arrow[u] & \ominus \arrow[l, dashed] \arrow[u] & a_1
\end{tikzcd}$$

Theorem characterizing concurrent strategies

Receptivity $\sigma : S \to A^\perp \parallel B$ is *receptive* when $\sigma(x) \subseteq^- y$ implies there is a *unique* $x' \in \mathcal{C}(S)$ such that $x \subseteq x'$ & $\sigma(x') = y$.

$$\begin{tikzcd} x \arrow[r, dotted, phantom, "\subseteq"] \arrow[d, mapsto] & x' \arrow[d, dotted, mapsto] \\ \sigma(x) \arrow[r, phantom, "\subseteq^-"] & y \end{tikzcd}$$

Innocence $\sigma : S \to A^\perp \parallel B$ is *innocent* when it is

$+$-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$ and

$-$-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

[$\rightarrowtriangle$ *stands for immediate causal dependency*]

Theorem Receptivity and innocence are necessary and sufficient for copy-cat to act as identity w.r.t. composition: $\sigma \odot \gamma_A \cong \sigma$ and $\gamma_B \odot \sigma \cong \sigma$ for all $\sigma : A \nrightarrow B$.

The bicategory of concurrent games

Definition A *strategy* is a receptive, innocent pre-strategy.

$\rightsquigarrow$ A bicategory, $\mathbf{Games}$, whose

objects are event structures with polarity—the games,

arrows are strategies $\sigma : A \nrightarrow B$

2-cells are maps of spans.

The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the functoriality of synchronized composition).

Strategies—alternative description 1

A strategy S in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that
(i) whenever $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$ there is a unique $x' \in \mathcal{C}(S)$ so that $x \subseteq x'$ & $\sigma x' = y$, *i.e.*

$$\begin{tikzcd} x \arrow[r, dotted, phantom, "\subseteq"] \arrow[d, mapsto, "\sigma"'] & x' \arrow[d, dotted, mapsto, "\sigma"] \\ \sigma x \arrow[r, phantom, "\subseteq^-"] & y \, , \end{tikzcd}$$

and
(ii) whenever $y \subseteq^+ \sigma x$ in $\mathcal{C}(A)$ there is a (necessarily unique) $x' \in \mathcal{C}(S)$ so that $x' \subseteq x$ & $\sigma x' = y$, *i.e.*

$$\begin{tikzcd} x' \arrow[r, dotted, phantom, "\subseteq"] \arrow[d, dotted, mapsto, "\sigma"'] & x \arrow[d, mapsto, "\sigma"] \\ y \arrow[r, phantom, "\subseteq^+"] & \sigma x \, . \end{tikzcd}$$

Strategies—alternative description 2

A strategy S in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that

(i) $\sigma x \overset{a}{\relbar\!\!\subset}$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{\relbar\!\!\subset}$ & $\sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$.

(ii)(+) If $x \overset{e}{\relbar\!\!\subset} x_1 \overset{e'}{\relbar\!\!\subset}$ & $pol_S(e) = +$ in $\mathcal{C}(S)$ and $\sigma x \overset{\sigma(e')}{\relbar\!\!\subset}$ in $\mathcal{C}(A)$, then $x \overset{e'}{\relbar\!\!\subset}$ in $\mathcal{C}(S)$.

(ii)(−) If $x \overset{e}{\relbar\!\!\subset} x_1 \overset{e'}{\relbar\!\!\subset}$ & $pol_S(e') = -$ in $\mathcal{C}(S)$ and $\sigma x \overset{\sigma(e')}{\relbar\!\!\subset}$ in $\mathcal{C}(A)$, then $x \overset{e'}{\relbar\!\!\subset}$ in $\mathcal{C}(S)$.

Notation $x \overset{e}{\relbar\!\!\subset} y$ iff $x \cup \{e\} = y$ & $e \notin x$, for configurations x, y, event e.
$x \overset{e}{\relbar\!\!\subset}$ iff $\exists y.\ x \overset{e}{\relbar\!\!\subset} y$.

Strategies—alternative description 3, via just $+$-moves

A strategy $\sigma : S \to A$ determines

$$\begin{tikzcd}
S \arrow[r, "q"'] \arrow[d, "\sigma"'] \arrow[dr, phantom, "{}^{-}\!\supseteq" near start] & S^{+} \arrow[dl, dotted, "d"] \\
A &
\end{tikzcd}$$

where q is projection and $d : \mathcal{C}(S) \to \mathcal{C}(A)$ s.t. $d(x) = \sigma[x]$. Universal property showing d determines σ:

$$\begin{tikzcd}
U \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "{}^{-}\!\supseteq" near start] & S^{+} \arrow[dl, dotted, "d"] \\
A &
\end{tikzcd}
\Rightarrow \exists!\phi \text{ s.t. }
\begin{tikzcd}
U \arrow[r, dashed, "\phi"] \arrow[rr, bend left, "g"] \arrow[dr, bend right, "f"'] & S \arrow[r, "q"'] \arrow[d, "\sigma"'] \arrow[dr, phantom, "{}^{-}\!\supseteq" near start] & S^{+} \arrow[dl, dotted, "d"] \\
& A &
\end{tikzcd}
\;\&\; \sigma\phi = f \;\&\; q\phi = g.$$

Deterministic strategies

Say an event structures with polarity S is **deterministic** iff

$$\forall X \subseteq_{\mathrm{fin}} S.\ \mathit{Neg}[X] \in \mathrm{Con}_S \Rightarrow X \in \mathrm{Con}_S \,,$$

where $\mathit{Neg}[X] =_{\mathrm{def}} \{s' \in S \mid \exists s \in X.\ \mathit{pol}_S(s') = - \ \&\ s' \leq s\}$.
Say a strategy $\sigma : S \to A$ is deterministic if S is deterministic.

Proposition An event structure with polarity S is deterministic iff $x \overset{s}{\relbar\!\!\!\subset}\ \&\ x \overset{s'}{\relbar\!\!\!\subset}\ \&\ \mathit{pol}_S(s) = +$ implies $x \cup \{s, s'\} \in \mathcal{C}(S)$, for all $x \in \mathcal{C}(S)$.

Notation $x \overset{e}{\relbar\!\!\!\subset} y$ iff $x \cup \{e\} = y \ \&\ e \notin x$, for configurations x, y, event e.
$x \overset{e}{\relbar\!\!\!\subset}$ iff $\exists y.\ x \overset{e}{\relbar\!\!\!\subset} y$.

Nondeterministic copy-cats

(i) Take A to consist of two $+$ve events and one $-$ve event, with any two but not all three events consistent. The construction of $\mathbb{C}\mathbb{C}_A$:

$$\begin{array}{ccccc} & \ominus & \rightarrowtriangle & \oplus & \\ A^\perp & \ominus & \rightarrowtriangle & \oplus & A \\ & \oplus & \leftarrowtriangle & \ominus & \end{array}$$

(ii) Take A to consist of two events, one $+$ve and one $-$ve event, inconsistent with each other. The construction $\mathbb{C}\mathbb{C}_A$:

$$\begin{array}{ccccc} A^\perp & \ominus & \rightarrowtriangle & \oplus & A \\ & \oplus & \leftarrowtriangle & \ominus & \end{array}$$

Lemma Let A be an event structure with polarity. The copy-cat strategy γ_A is deterministic iff A satisfies

$$\forall x \in \mathcal{C}(A).\ x \overset{a}{\longrightarrow\!\!\subset} \ \&\ x \overset{a'}{\longrightarrow\!\!\subset} \ \&\ pol_A(a) = + \ \&\ pol_A(a') = - \\ \Rightarrow x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad (\ddagger)$$

Lemma The composition $\tau \odot \sigma$ of two deterministic strategies σ and τ is deterministic.

Lemma A deterministic strategy $\sigma : S \to A$ is mono (equivalently, injective on configurations).

$\rightsquigarrow$ sub-bicategory $\mathbf{DGames}$, equivalent to an order-enriched category; a characterization of deterministic strategies as certain subfamilies.

Related work

Stable spans, profunctors and stable functions The sub-bicategory of $\mathbf{Games}$ where the events of games are purely $+$ve is equivalent to the bicategory of stable spans:

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & & \longleftrightarrow & & S^+ \arrow[dl, "\sigma_1^-"'] \arrow[dr, "\sigma_2^+"] & \\ A^\perp & & B & & A & & B\,, \end{tikzcd}$$

where S^+ is the projection of S to its $+$ve events; σ_2^+ is the restriction of σ_2 to S^+ is rigid; σ_2^- is a *demand map* taking $x \in \mathcal{C}(S^+)$ to $\sigma_1^-(x) = \sigma_1[x]$.
Composition of stable spans coincides with composition of their associated profunctors.
When deterministic (and event structures are countable) we obtain a sub-bicategory equivalent to Berry's **dI-domains and stable functions**.

Related work continued

Ingenuous strategies Deterministic concurrent strategies coincide with the *receptive ingenuous* strategies of Melliès and Mimram.

Closure operators A deterministic strategy $\sigma : S \to A$ determines a closure operator φ on $\mathcal{C}^\infty(S)$: for $x \in \mathcal{C}^\infty(S)$,

$$\varphi(x) = x \cup \{s \in S \mid pol(s) = + \ \&\ Neg[\{s\}] \subseteq x\}\,.$$

The closure operator φ on $\mathcal{C}^\infty(S)$ induces a *partial* closure operator φ_p on $\mathcal{C}^\infty(A)$ and in turn a closure operator $\varphi_p^\top$ on $\mathcal{C}^\infty(A)^\top$—Abramsky and Melliès' concurrent strategies .

Simple games *"Simple games"* [Hyland $et\ al.$] arise when we restrict $\mathbf{Games}$ to objects and deterministic strategies which are 'tree-like'—alternating polarities, with conflicting branches, beginning with opponent moves.

Adjunctions between games

$$
\begin{tikzcd}
 & \mathcal{PA}_r \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] \arrow[d, shift right] \arrow[d, phantom, "\vdash"] & \mathcal{PF}_r \arrow[l, shift right] \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] \arrow[d, shift right] \arrow[d, phantom, "\vdash"] & \mathcal{PE}_r \arrow[l, shift right] \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] & \mathcal{PE}_t \arrow[l, shift right] \\
\mathcal{PA}_t^{-\#} \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] & \mathcal{PA}_t^{\#} \arrow[l, shift right] \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] \arrow[u, hook, shift right] & \mathcal{PF}_t^{\#} \arrow[l, shift right] \arrow[u, hook, shift right] & &
\end{tikzcd}
$$

Conway games inhabit $\mathcal{PF}_t^{\#} = \mathcal{PF}_r^{\#}$, a coreflective subcategory of $\mathcal{PE}_t$. Conway's 'sum' is obtained by applying the right adjoint to their $\|$-composition in $\mathcal{PE}_t$.

'Simple games' belong to $\mathcal{PA}_r^{-\#}$, "polarized" games, starting with moves of Opponent. 'Tensor' of simple games got by applying the right adjoint of $\mathcal{PA}_t^{-\#} \hookrightarrow \mathcal{PE}_t$ to their $\|$-composition in $\mathcal{PE}_t$.

Extensions

- To games with backtracking, via copying. $E.g.$, obtain the (co)monads for Hyland-Ong games from (co)monads on event structures with symmetry.
- To adjunctions between games and strategies, from the present adjunctions between games.
- To games with *winning conditions* and *winning strategies*. ✓
- To other (algorithmically amenable) models.
  [Can exploit the central position of event structures amongst such models]
- To games with stochastic/probabilistic structure ($very\ preliminary$).
  [Relies on concurrent strategies being nondeterministic]