# A tetrachotomy for positive first-order logic without equality

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# Model Checking problem

We are interested in the parameterisation of the model checking problem by the model. Fix a logic $\mathscr{L}$ and fix $\mathcal{D}$.

The problem "$\mathscr{L}(\mathcal{D})$" has

- Input: a sentence $\varphi$ of $\mathscr{L}$.
- Question: does $\mathcal{D} \models \varphi$?

We consider syntactic fragments $\mathscr{L}$ of FO and structures $\mathcal{D}$ that are relational and finite.

# Complexity of Model Checking

| Fragment | Dual | Classification? |
|---|---|---|
| $\{\exists, \vee\}$<br>$\{\exists, \vee, =\}$ | $\{\forall, \wedge\}$<br>$\{\forall, \wedge, \neq\}$ | Logspace |
| $\{\exists, \wedge, \vee\}$<br>$\{\exists, \wedge, \vee, =\}$ | $\{\forall, \wedge, \vee\}$<br>$\{\forall, \wedge, \vee, \neq\}$ | Logspace if there is some element $a$ s.t. all relations are $a$-valid, an |
| $\{\exists, \wedge\}$<br>$\{\exists, \wedge, =\}$ | $\{\forall, \vee\}$<br>$\{\forall, \vee, \neq\}$ | CSP dichotomy conjecture: P or NP-complete |
| $\{\exists, \wedge, \neq\}$ | $\{\forall, \vee, =\}$ | NP-complete for $\lvert\mathcal{D}\rvert \geq 3$, reduces to Schaefer classes otherwise. |
| $\{\exists, \forall, \wedge\}$<br>$\{\exists, \forall, \wedge, =\}$ | $\{\exists, \forall, \vee\}$<br>$\{\exists, \forall, \vee, \neq\}$ | QCSP polychotomy: P, NP-complete, or Pspace-complete ? |
| $\{\exists, \forall, \wedge, \neq\}$ | $\{\exists, \forall, \vee, =\}$ | Pspace-complete for $\lvert\mathcal{D}\rvert \geq 3$, reduces to Schaefer classes for Quantified Sat otherwise. |
| $\{\forall, \exists, \wedge, \vee\}$ | | **Positive equality free: the rest of this talk** |
| $\{\forall, \exists, \wedge, \vee, =\}$ $\{\forall, \exists, \wedge, \vee, \neq\}$<br>$\{\neg, \exists, \forall, \wedge, \vee, =\}$ | | P when $\lvert\mathcal{D}\rvert \leq 1$, Pspace-complete otherwise |
| $\{\neg, \exists, \forall, \wedge, \vee\}$ | | P when $\mathcal{D}$ contains only empty or full relations, Pspace-complete otherwise |

- See B. Martin's paper on this for more details (CiE'08)

# Tetrachotomy for $\{\exists, \forall, \wedge, \vee\}$-FO

When $|\mathcal{D}| \leq 4$, we obtained a tetrachotomy between

Pspace-complete

NP-complete co-NP-complete

Logspace

Our approach was algebraic but direct : i.e. direct complexity classification in suitable finite lattices [LICS'09, CSL'10].

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Pspace-complete

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Logspace

Our approach was algebraic but direct : i.e. direct complexity classification in suitable finite lattices [LICS'09, CSL'10].

- It turns out that we knew the "tractable" cases.
- We complete the classification by proving that all other cases are Pspace-complete.

## Some Ingredients of our approach

- Galois Connection
- "Tractability" via relativisation of quantifiers

# Ferdinand Börner's tips for Galois Connections

| relation closed under | preserved by "operation" |
|---|---|
| absence of $\exists$ | partial |
| presence of $\forall$ | "surjective" |
| presence of $\vee$ | unary |
| presence of $=$<br>absence of $=$ | functions<br>hyperfunctions |
| presence of $\neq$ | injective |
| presence of atomic $\neg$ | full |

For $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}$, we will need to consider the surjective hyper endomorphisms of the structure $\mathcal{D}$.

# Surjective hyper endomorphisms

A *surjective hyper-operation* (shop) on a set $D$ is a function

$$f : D \to \mathcal{P}(D)$$

that satisfies

- for all $x \in D$, $f(x) \neq \emptyset$ (totality).
- for all $y \in D$, there exists $x \in D$ s.t. $y \in f(x)$ (surjectivity).

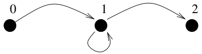
A *surjective hyper-endomorphism* (she) of $\mathcal{D}$ is a surjective hyper-operation $f$ on $D$ that preserves all extensional relations $R$ of $\mathcal{D}$,

- if $R(x_1, \ldots, x_i) \in \mathcal{D}$ then, for all $y_1 \in f(x_1), \ldots, y_i \in f(x_i)$, $R(y_1, \ldots, y_i) \in \mathcal{D}$.

# Example

preserves

| | |
|---|---|
| 0 | $\{0\}$ |
| 1 | $\{1\}$ |
| 2 | $\{1, 2\}$ |




does not preserve

| | |
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| 0 | $\{0\}$ |
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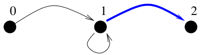


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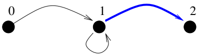


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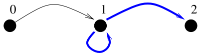


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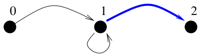


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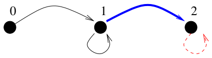


does not preserve

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# Monoid

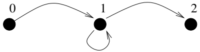

has the following set of surjective hyper endomorphisms:

$$\left\{ \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array}, \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array}, \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array} \right\}$$

which forms in fact a monoid:

$$\left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array} \right\rangle.$$

# Monoid




$\mathcal{D}$

has the following set of surjective hyper endomorphisms:

$$\mathrm{shE}(\mathcal{D}) = \left\{ \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array}, \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array}, \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array} \right\}$$

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$$\mathrm{shE}(\mathcal{D}) = \left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array} \right\rangle.$$

# Down Shop Monoid

- A set of shops on $D$ is a down-shop-monoid, if it contains $id_D$, and is closed under composition and sub-shops.

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The *identity* shop $id_S$ is defined by $x \mapsto \{x\}$.

# Down Shop Monoid

- A set of shops on $D$ is a down-shop-monoid, if it contains $id_D$, and is closed under composition and sub-shops.

Given shops $f$ and $g$, define the *composition* $g \circ f$ by

$$x \mapsto \{z : \exists y \; z \in g(y) \wedge y \in f(x)\}.$$

# Down Shop Monoid

- A set of shops on $D$ is a down-shop-monoid, if it contains $id_D$, and is closed under composition and sub-shops.

A shop $f$ is a *sub-shop* of $g$ if $f(x) \subseteq g(x)$, for all $x$.

# Down Shop Monoid

- A set of shops on $D$ is a down-shop-monoid, if it contains $id_D$, and is closed under composition and sub-shops.
- We write $\langle F \rangle$ for the down-shop-monoid generated by a set of surjective hyper-operations $F$.

# A suitable Galois Connection

**Theorem (Madelaine, Martin '09)**

*A relation is* $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}$*-expressible in a finite structure* $\mathcal{D}$*, if and only if, it is invariant under the surjective hyper endomorphisms of* $\mathcal{D}$*.*

# A suitable Galois Connection

Let shE($\mathcal{D}$) be the set of surjective hyper-endomorphisms of a structure $\mathcal{D}$.

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*For finite $\mathcal{D}$ and $\mathcal{D}'$ (s.t. $D = D'$),*
$\text{shE}(\mathcal{D}) \subseteq \text{shE}(\mathcal{D}') \Rightarrow$
$\{\exists, \forall, \wedge, \vee\}\text{-}\mathrm{FO}(\mathcal{D}') \leq_{\text{Logspace}} \{\exists, \forall, \wedge, \vee\}\text{-}\mathrm{FO}(\mathcal{D})$.

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Motto. surjective hyper-endomorphisms control expressive power and complexity.

# A suitable Galois Connection

Let $\mathrm{shE}(\mathcal{D})$ be the set of surjective hyper-endomorphisms of a structure $\mathcal{D}$.
If $F$ is a set of surjective hyper-operations then $\mathrm{Inv}(F)$ is the set of relations of which $F$ are surjective hyper-endomorphisms.

### Theorem (Madelaine, Martin '10)

*For a finite structure $\mathcal{D}$ and a set of shops $F$, the following holds,*

- $\langle \mathcal{D} \rangle_{\{\exists,\forall,\wedge,\vee\}\text{-FO}} = \mathrm{Inv}(\mathrm{shE}(\mathcal{D}))$; *and,*
- $\langle F \rangle = \mathrm{shE}(\mathrm{Inv}(F))$.

# Surjective hyper operations of special interest

Let $D$ be a finite set with elements $c, d$. We define the following types of surjective hyper operations.

$$A_c(x) := \begin{cases} D & \text{if } x = c \\ \{?\} & \text{otherwise.} \end{cases}$$

e.g.

| | |
|---|---|
| 0 | 0 |
| 1 | 3 |
| 2 | 0123 |
| 3 | 12 |

i.e. $A_c(c) = D$

$$E_c(x) := \{?, c\}$$

e.g.

| | |
|---|---|
| 0 | 012 |
| 1 | 1 |
| 2 | 12 |
| 3 | 13 |

i.e. $E_c{}^{-1}(c) = D$

$$\forall\exists_{c,d}(x) := \begin{cases} D & \text{if } x = c \\ \{d\} & \text{otherwise.} \end{cases}$$

e.g.

| | |
|---|---|
| 0 | 0123 |
| 1 | 2 |
| 2 | 2 |
| 3 | 2 |

# Quantifier Elimination

$$A_c(c) = D \qquad {E_d}^{-1}(d) = D$$

$$\forall\exists_{c,d}(c) = D \qquad {\forall\exists_{c,d}}^{-1}(d) = D$$

| presence of | complexity drops to | “algorithm” |
|---|---|---|
| $A_c$ | NP | evaluate all $\forall$ to $c$ |
| $E_d$ | co-NP | evaluate all $\exists$ to $d$ |
| $\forall\exists_{c,d}$ | Logspace | simultaneously do both |

- We shall see that these special surjective hyper operations characterise fully the complexity.
- For example, if a relational structure $\mathcal{D}$ is preserved by an $A$-shop but no $\forall\exists$-shop, the model checking problem $\{\exists, \forall, \wedge, \vee\}\text{-FO}(\mathcal{D})$ is NP-complete

# Warm-up: the boolean case

There are five monoids in this case.

$$\begin{tikzcd}
 & \left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 01 \end{array} \right\rangle & \\
\left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 01 \end{array} \right\rangle \arrow[ur] & \left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array} \right\rangle \arrow[u] & \left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \end{array} \right\rangle \arrow[ul] \\
 & \left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \end{array} \right\rangle \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

**Theorem**

*If* shE($\mathcal{D}$) *is green above, then* $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\mathcal{D})$ *is in* Logspace; *otherwise it is* Pspace*-complete.*

# The three-element case

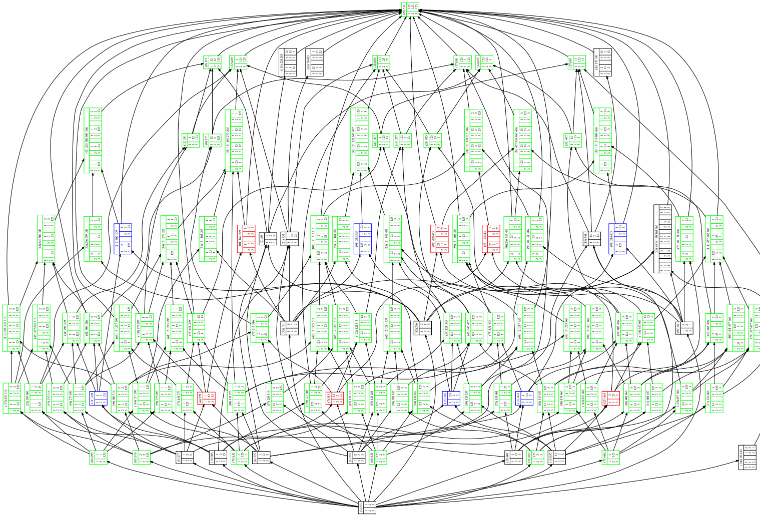
The lattice is considerably richer. The problem class $\{\exists, \forall, \wedge, \vee\}$-$\textsc{fo}(\mathcal{D})$ displays tetrachotomy, between

Logspace

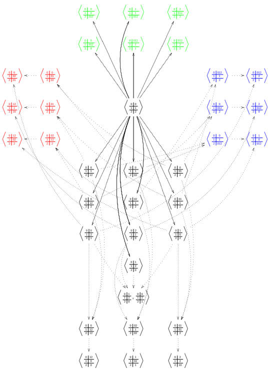
NP-complete co-NP-complete

Pspace-complete

# lattice in the 3 element case



Most of these are green "L" cases. The bottom of the lattice is

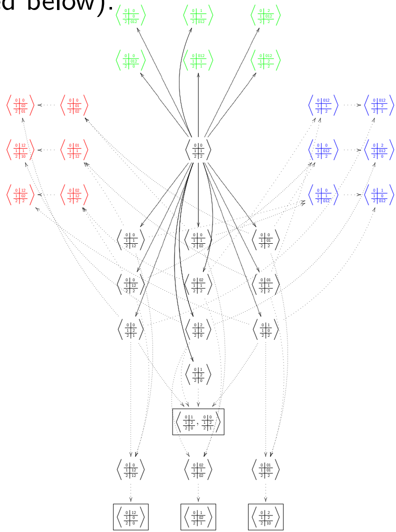


### Theorem (Madelaine & Martin 2009)

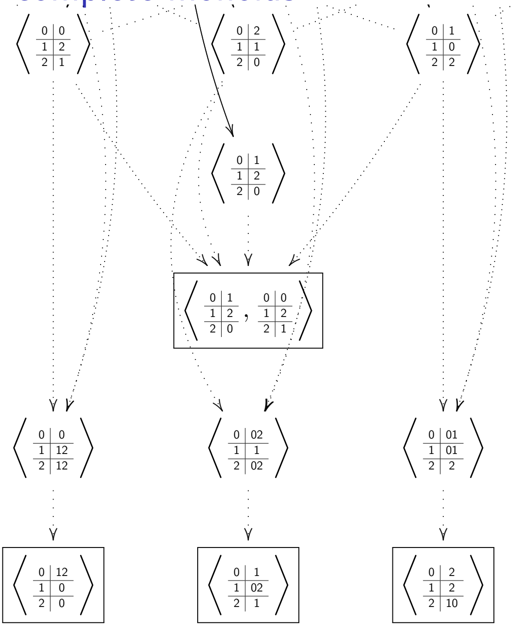
*If* shE($\mathcal{D}$) *is green, blue or red, above, then* $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\mathcal{D})$ *is in* L, *is* NP-*complete or is* co-NP-*complete, respectively; otherwise it is* Pspace-*complete.*

# Maximal Pspace-complete monoids

There are four maximal Pspace-complete monoids in the 3 element case (drawn boxed below).



# Maximal Pspace-complete monoids



# Maximal Pspace-complete monoids

There are 20 maximal Pspace-complete monoids in the 4 element case.

Class I

$$\left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \\ \hline 2 & 012 \\ \hline 3 & 013 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 013 \\ \hline 3 & 012 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 2 \\ \hline 1 & 012 \\ \hline 2 & 0 \\ \hline 3 & 023 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 023 \\ \hline 2 & 2 \\ \hline 3 & 012 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 3 \\ \hline 1 & 013 \\ \hline 2 & 023 \\ \hline 3 & 0 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 023 \\ \hline 2 & 013 \\ \hline 3 & 3 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 012 \\ \hline 1 & 2 \\ \hline 2 & 1 \\ \hline 3 & 123 \end{array}, \begin{array}{c|c} 0 & 123 \\ \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 012 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 013 \\ \hline 1 & 3 \\ \hline 2 & 123 \\ \hline 3 & 1 \end{array}, \begin{array}{c|c} 0 & 123 \\ \hline 1 & 1 \\ \hline 2 & 013 \\ \hline 3 & 3 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 023 \\ \hline 1 & 123 \\ \hline 2 & 3 \\ \hline 3 & 2 \end{array}, \begin{array}{c|c} 0 & 123 \\ \hline 1 & 023 \\ \hline 2 & 2 \\ \hline 3 & 3 \end{array} \right\rangle$$

Class II

$$\left\langle \begin{array}{c|c} 0 & 23 \\ \hline 1 & 23 \\ \hline 2 & 01 \\ \hline 3 & 01 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 13 \\ \hline 1 & 02 \\ \hline 2 & 13 \\ \hline 3 & 02 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 03 \\ \hline 2 & 03 \\ \hline 3 & 12 \end{array} \right\rangle$$

Class III

$$\left\langle \begin{array}{c|c} 0 & 3 \\ \hline 1 & 3 \\ \hline 2 & 3 \\ \hline 3 & 012 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 2 \\ \hline 1 & 2 \\ \hline 2 & 013 \\ \hline 3 & 2 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 023 \\ \hline 2 & 1 \\ \hline 3 & 1 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 123 \\ \hline 1 & 0 \\ \hline 2 & 0 \\ \hline 3 & 0 \end{array} \right\rangle$$

Class IV

$$\left\langle \begin{array}{c|c} 0 & 2 \\ \hline 1 & 2 \\ \hline 2 & 01 \\ \hline 3 & 3 \end{array}, \begin{array}{c|c} 0 & 01 \\ \hline 1 & 01 \\ \hline 2 & 3 \\ \hline 3 & 2 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 02 \\ \hline 2 & 1 \\ \hline 3 & 3 \end{array}, \begin{array}{c|c} 0 & 02 \\ \hline 1 & 3 \\ \hline 2 & 02 \\ \hline 3 & 1 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 03 \\ \hline 2 & 2 \\ \hline 3 & 1 \end{array}, \begin{array}{c|c} 0 & 03 \\ \hline 1 & 2 \\ \hline 2 & 1 \\ \hline 3 & 03 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 12 \\ \hline 1 & 0 \\ \hline 2 & 0 \\ \hline 3 & 3 \end{array}, \begin{array}{c|c} 0 & 3 \\ \hline 1 & 12 \\ \hline 2 & 12 \\ \hline 3 & 0 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 13 \\ \hline 1 & 0 \\ \hline 2 & 2 \\ \hline 3 & 0 \end{array}, \begin{array}{c|c} 0 & 2 \\ \hline 1 & 13 \\ \hline 2 & 0 \\ \hline 3 & 13 \end{array} \right\rangle$$

$$\left\langle \begin{array}{c|c} 0 & 23 \\ \hline 1 & 1 \\ \hline 2 & 0 \\ \hline 3 & 0 \end{array}, \begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \\ \hline 2 & 23 \\ \hline 3 & 23 \end{array} \right\rangle$$

Class V

$$\left\langle \begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \\ \hline 2 & 2 \\ \hline 3 & 3 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 2 \\ \hline 2 & 1 \\ \hline 3 & 3 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 3 \\ \hline 3 & 2 \end{array} \right\rangle$$

The hard part is in proving there are no others.

# Limitation of the “classification by lattice” method

| Domain | Classification | Method | Maximally hard monoids |
|---|---|---|---|
| 2 | done | by hand | 1 |
| 3 | done | by hand, computer checked | 4 |
| 4 | done | by computer | 20 |
| 5 | failed attempt | by computer | 161 |

Stuck. We need to move away from the lattice.

# Tetrachotomy for all finite domains

## Theorem (Madelaine & Martin 2011)

*Let $\mathcal{D}$ be any finite structure.*

I. *If* shE($\mathcal{D}$) *contains both an* A-*shop and an* E-*shop, then* $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\mathcal{D})$ *is in* Logspace.

II. *If* shE($\mathcal{D}$) *contains an* A-*shop but no* E-*shop, then* $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\mathcal{D})$ *is* NP-*complete.*

III. *If* shE($\mathcal{D}$) *contains an* E-*shop but no* A-*shop, then* $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\mathcal{D})$ *is* co-NP-*complete.*

IV. *If* shE($\mathcal{D}$) *contains neither an* A-*shop nor an* E-*shop, then* $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\mathcal{D})$ *is in* Pspace-*complete.*

Proved for domain size 2,3,4 (using the lattice).

Settled for larger domains (without the lattice).

# Ingredients of our approach

Previous ingredients:

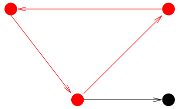
- Galois Connection
- "Tractability" via relativisation of quantifiers

New ingredients:

- A suitable notion of core for $\{\exists, \forall, \wedge, \vee\}$-FO
- Normal form for the monoid associated with the core of a structure $\mathcal{D}$
- Generic hardness proof

# Core

For $\mathrm{CSP}$ there is the well-established notion of core. The core of a structure $\mathcal{D}$ is a minimal induced substructure $\mathcal{X} \subseteq \mathcal{D}$ all of whose endomorphisms are automorphisms.



It is well-known that $\mathcal{X}$ is unique and $\mathrm{CSP}(\mathcal{D}) = \mathrm{CSP}(\mathcal{X})$.

# Core and relativisation

Another way to define the core is as a minimal subset $X \subseteq D$ such that for all positive conjunctive $\phi(\overline{x})$:

$$\mathcal{D} \models \exists \overline{x}\ \phi(\overline{x}) \quad \text{iff} \quad \mathcal{D} \models \exists \overline{x} \in X\ \phi(\overline{x}).$$

# Core and relativisation

Another way to define the core is as a minimal subset $X \subseteq D$ such that for all positive conjunctive $\phi(\overline{x})$:

$$\mathcal{D} \models \exists \overline{x}\ \phi(\overline{x}) \ \text{ iff } \ \mathcal{D} \models \exists \overline{x} \in X\ \phi(\overline{x}).$$

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Does there exist a "core"-like notion for $\{\exists, \forall, \wedge, \vee\}$-FO?

Yes.
But we need 2 relativising sets $U$ (universal) and $X$ (existential).

# $U$-$X$-core

**Theorem (Madelaine & Martin, 2011)**

*The following are equivalent*

1. *There is* $f \in \mathrm{shE}(\mathcal{D})$ *s.t.* $f(U) = D$ *and* $f^{-1}(X) = D$
2. *for all positive equality-free* $\phi$, $\mathcal{D} \models \phi \Leftrightarrow \mathcal{D} \models \phi_{[\forall/U, \exists/X]}$.

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We may minimise $X$ and $U$, then maximise their intersection to obtain a monoid we call reduced.

The substructure of $\mathcal{D}$ induced by $U \cup X$ satisfies the same sentences of $\{\exists, \forall, \wedge, \vee\}$-FO as $\mathcal{D}$. We call it the $U - X$-**core** (as it is unique up to isomorphism).

# Example of a reduced monoid

Consider the domain 5 maximal monoid $\left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 0234 \\ \hline 2 & 024 \\ \hline 3 & 0124 \\ \hline 4 & 4 \end{array}, \begin{array}{c|c} 0 & 4 \\ \hline 1 & 0124 \\ \hline 2 & 024 \\ \hline 3 & 0234 \\ \hline 4 & 0 \end{array} \right\rangle$.

$$\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0234 \\ \hline 2 & 024 \\ \hline 3 & 0124 \\ \hline 4 & 0 \end{array}, \quad \begin{array}{c|c} 0 & 0 \\ \hline 1 & 0234 \\ \hline 2 & 024 \\ \hline 3 & 0124 \\ \hline 4 & 44 \end{array} \qquad U := \{1,3\} \text{ and } X := \{0,4\}.$$

Thus we are equivalent to the reduced monoid

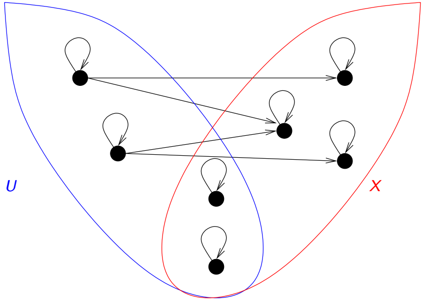
$$\left\langle \begin{array}{c|c} 0 & 0 \\ \hline 1 & 034 \\ \hline 3 & 014 \\ \hline 4 & 4 \end{array}, \begin{array}{c|c} 0 & 4 \\ \hline 1 & 014 \\ \hline 3 & 034 \\ \hline 4 & 0 \end{array} \right\rangle.$$

# Tractable cases

| Case | Complexity | $A$-shop | $E$-shop | $U$-$X$-core | Relativises into | Dual |
|---|---|---|---|---|---|---|
| I | Logspace | yes | yes | $\lvert U \rvert = 1, \lvert X \rvert = 1$ | $\{\wedge, \vee\}$-FO | I |
| II | NP-complete | yes | no | $\lvert U \rvert = 1, \lvert X \rvert \geq 2$ | $\{\exists, \wedge, \vee\}$-FO | III |
| III | co-NP-complete | no | yes | $\lvert U \rvert \geq 2, \lvert X \rvert = 1$ | $\{\forall, \vee, \wedge\}$-FO | II |

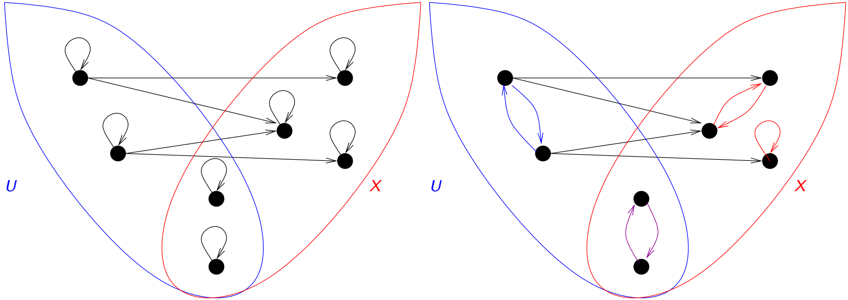
Remaining case. when both $|U| \geq 2$ and $|X| \geq 2$.

# Canonical shop and normal form of the reduced monoid




Canonical shop

# Canonical shop and normal form of the reduced monoid




Canonical shop

3-permuted form

All shops in the reduced monoid are in a similar form up to permutation of $U \cap X$, $X \setminus U$ and $U \setminus X$, or sub-shops thereof.

# Pspace-hardness

$U$ and $X$ have both size at least 2.

We consider three cases:

- $U = X$.
- $U \neq X$ and $U \cap X \neq \emptyset$.
- $U \cap X = \emptyset$.

# Pspace-hardness

$U$ and $X$ have both size at least 2.

We consider three cases:

- $U = X$.
  - shops are necessarily "permutations".
  - We know from previous results that this case is Pspace-complete.
- $U \neq X$ and $U \cap X \neq \emptyset$.
- $U \cap X = \emptyset$.

# Pspace-hardness

$U$ and $X$ have both size at least 2.

We consider three cases:

- $U = X$.
- $U \neq X$ and $U \cap X \neq \emptyset$.
  - one set can not be included in another.
  - We complete the monoid by adding more shops to blur $U \cap X$ to a single element and $U \Delta X$ to a single element.
  - This amounts to consider a Pspace-hard monoid from the 2-element case.
- $U \cap X = \emptyset$.

# Pspace-hardness

$U$ and $X$ have both size at least 2.

We consider three cases:

- $U = X$.
- $U \neq X$ and $U \cap X \neq \emptyset$.
- $U \cap X = \emptyset$.
  - we are unable to exhibit such a simple proof.
  - we complete the monoid by adding all shops in the 3-permuted form.
  - thanks to the relative simplicity of this completed monoid, we can provide a generic hardness proof inspired from the 4 element case

# Tetrachotomy for all domains

| Tetrachotomy for $\{\exists, \forall, \land, \lor\}$-FO($\mathcal{D}$) | | | | | | |
|---|---|---|---|---|---|---|
| Case | Complexity | $A$-shop | $E$-shop | $U$-$X$-core | Relativises into | Dual |
| I | Logspace | yes | yes | $\lvert U\rvert = 1$, $\lvert X\rvert = 1$ | $\{\land, \lor\}$-FO | I |
| II | NP-complete | yes | no | $\lvert U\rvert = 1$, $\lvert X\rvert \geq 2$ | $\{\exists, \land, \lor\}$-FO | III |
| III | co-NP-complete | no | yes | $\lvert U\rvert \geq 2$, $\lvert X\rvert = 1$ | $\{\forall, \lor, \land\}$-FO | II |
| IV | Pspace-complete | no | no | $\lvert U\rvert \geq 2$, $\lvert X\rvert \geq 2$ | $\{\exists, \forall, \lor, \land\}$-FO | IV |

Bonus. A notion of core for quantified constraints.

# The meta problem is NP-complete.

The $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\sigma)$ meta-problem takes as input a finite $\sigma$-structure $\mathcal{D}$ and answers L, NP-complete, co-NP-complete or Pspace-complete, according to the complexity of $\{\exists, \forall, \wedge, \vee\}$-$\textsc{fo}(\mathcal{D})$.
It is NP-hard even for some fixed and finite signature $\sigma_0$.

# Conclusion

| Fragment | Dual | Classification? |
|---|---|---|
| $\{\exists, \wedge\}$<br>$\{\exists, \wedge, =\}$ | $\{\forall, \vee\}$<br>$\{\forall, \vee, \neq\}$ | CSP Dichotomy conjecture (P or NP-complete). solved for (undirected) graphs (Hell & Nešetřil), in the boolean case (Schaefer), the 3 element case (Bulatov) and the conservative case (Bulatov, Barto). |
| $\{\exists, \forall, \wedge\}$<br>$\{\exists, \forall, \wedge, =\}$ | $\{\exists, \forall, \vee\}$<br>$\{\exists, \forall, \vee, \neq\}$ | P/Pspace-complete dichotomy in the boolean case (Schaefer). In general, no precise conjecture. Partial results exhibit P, NP-complete, and Pspace-complete complexities: via the algebraic approach by Chen *et. al.* or a combinatorial approach for graphs and digraphs (Madelaine & Martin). Even the case of (undirected) graphs remains open. |
| $\{\forall, \exists, \wedge, \vee\}$ | | Tetrachotomy |