QC*

Achieving cut, deduction, and other properties with a variation on quasi-classical logic

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Introduction

- Contradictory information is the norm in large systems
- Classical logic is explosive
  I.E. $\{\alpha, \neg\alpha\} \vdash \beta$, for any sentences α, β.
- Classical logic unsuitable for reasoning under contradictory information
- Paraconsistent logics avoid explosion

Introduction

- Quasi-classical logic (QC) [BH95, Hun99] is a paraconsistent logic that:
  - Is not explosive.
  - Preserves the usual boolean equivalences.
  - Does not require consistency checking.
  - Supports disjunctive syllogism.

.

- QC fails at some intuitive properties

  E.G. If $\Gamma \mid\!\sim \alpha$ and $\Gamma \mid\!\sim \alpha \to \beta$ then $\Gamma \mid\!\sim \beta$ (Right Modus Ponens)
- We tweak QC, obtain variant QC*

Basic definitions

Definition (Language)

Let the language $\mathcal{L}$ be the set of classical propositional formulas over atoms $\mathcal{A}$ and connectives $\wedge, \vee, \rightarrow, \neg$.

Definition (Literal)

A *literal* is an atom or the negation of an atom.

Definition (Clause)

A *clause* is a finite set of literals (disjunctive).

Resolution

Definition (Clauses of a theory)

Each $\Gamma \subseteq \mathcal{L}$ can be converted into a set of clauses, denoted as Clauses(Γ).

Definition (Binary resolution)

$$\frac{\{\lambda, \lambda_{1,1}, \ldots, \lambda_{1,n}\} \qquad \{\neg\lambda, \lambda_{2,1}, \ldots, \lambda_{2,m}\}}{\{\lambda_{1,1}, \ldots, \lambda_{1,n}, \lambda_{2,1}, \ldots, \lambda_{2,m}\}}$$

Definition (Resolvents)

Resolvents(Φ) is the closure of Φ under binary resolution (Φ a set of clauses)

Quasi-classical logic QC

Definition (QC consequence)

For $\Gamma \subseteq \mathcal{L}$ and $\phi \in \mathcal{L}$,
$\Gamma \vdash_{\mathsf{QC}} \phi$ if and only if[1]

- For every $C \in \text{Clauses}(\phi)$,
- There exists $B \in \text{Resolvents}(\text{Clauses}(\Gamma))$ such that
  - $B \neq \{\}$ and
  - $B \subseteq C$.

E.G. $\{p\} \vdash_{\mathsf{QC}} p$

E.G. $\{\neg p\} \vdash_{\mathsf{QC}} \neg p \vee q$

E.G. $\{\neg p\} \vdash_{\mathsf{QC}} p \to q$

E.G. $\{p, \neg p \vee q\} \vdash_{\mathsf{QC}} q$

E.G. $\{p, p \to q\} \vdash_{\mathsf{QC}} q$

E.G. $\{p, \neg p\} \nvdash_{\mathsf{QC}} q$

[1] For a Fitch-style proof theory, see [Hun99].

Failed properties

QC fails at several well-known properties of classical logic examined in [Hun99].

For $\Gamma, \Delta \in \mathcal{P}(\mathcal{L}), \alpha, \beta \in \mathcal{L}$,

- Right Modus Ponens (RMP)

$$\text{If } \Gamma \mid\!\sim \alpha \text{ and } \Gamma \mid\!\sim \alpha \rightarrow \beta \text{ then } \Gamma \mid\!\sim \beta$$

E.G. $\{p, \neg p\} \vdash_{QC} p$ and $\{p, \neg p\} \vdash_{QC} p \rightarrow q$, but $\{p, \neg p\} \nvdash_{QC} q$

- Deduction (Ded)

$$\text{If } \Gamma \mid\!\sim \alpha \rightarrow \beta \text{ then } \Gamma \cup \{\alpha\} \mid\!\sim \beta$$

- Unit Cumulativity (UCu)

$$\text{If } \Gamma \cup \{\alpha\} \mid\!\sim \beta \text{ and } \Gamma \mid\!\sim \alpha \text{ then } \Gamma \mid\!\sim \beta$$

- Cut

$$\text{If } \Gamma \cup \{\alpha\} \mid\!\sim \beta \text{ and } \Delta \mid\!\sim \alpha \text{ then } \Gamma \cup \Delta \mid\!\sim \beta$$

QC + {Cut or Ded or RMP or UCu } is explosive

$\{p, \neg p\} \vdash_{\mathsf{QC}} p$ and
$\{p, \neg p\} \vdash_{\mathsf{QC}} p \to q$,
but $\{p, \neg p\} \nvdash_{\mathsf{QC}} q$

Theorem

If a consequence relation $\mid\!\sim$ is at least as strong as $\vdash_{\mathsf{QC}}$ and satisfies RMP (or Ded or Cut or UCu) then $\mid\!\sim$ is explosive.

I.E. QC MUST fail at Cut, Ded, RMP, and UCu.

Culprit

QC suports unrestricted $\vee$-introduction.
$\{\neg p\} \vdash_{QC} p \rightarrow q$

$$[\vee\text{-introduction}]\ \frac{\alpha}{\alpha \vee \beta}$$

In QC*, restrict the $\vee$-introduction rule[2]

$$[\text{Restricted } \vee\text{-introduction}]\ \frac{\alpha \qquad \beta}{\alpha \vee \beta}$$

[2] The idea of restricting disjunction introduction in this way appears in [Hew08].

Quasi-classical logic variant QC*

Definition (QC* consequence)

For $\Gamma \subseteq \mathcal{L}$ and $\phi \in \mathcal{L}$,
$\Gamma \vdash_{QC*} \phi$ if and only if[3]

- For each $C \in \text{Clauses}(\phi)$,
- There exists $B_1, \ldots, B_k \in \text{Resolvents}(\text{Clauses}(\Gamma))$ such that
  - $C = \bigcup_i B_i$

E.G. $\{p\} \vdash_{QC*} p$

E.G. $\{\neg p\} \nvdash_{QC*} \neg p \vee q$

E.G. $\{\neg p\} \nvdash_{QC*} p \rightarrow q$

E.G. $\{p, \neg p \vee q\} \vdash_{QC*} q$

E.G. $\{p, p \rightarrow q\} \vdash_{QC*} q$

E.G. $\{p, \neg p\} \nvdash_{QC*} q$

[3] For a Fitch-style proof theory, see [Kao11]

Succeeding

For $\Gamma, \Delta \in \mathcal{P}(\mathcal{L}), \alpha, \beta \in \mathcal{L}$,

- Right Modus Ponens (RMP)

$$\text{If } \Gamma \vdash_{\mathsf{QC*}} \alpha \text{ and } \Gamma \vdash_{\mathsf{QC*}} \alpha \to \beta \text{ then } \Gamma \vdash_{\mathsf{QC*}} \beta$$

- Deduction (Ded)

$$\text{If } \Gamma \vdash_{\mathsf{QC*}} \alpha \to \beta \text{ then } \Gamma \cup \{\alpha\} \vdash_{\mathsf{QC*}} \beta$$

- Unit Cumulativity (UCu)

$$\text{If } \Gamma \cup \{\alpha\} \vdash_{\mathsf{QC*}} \beta \text{ and } \Gamma \vdash_{\mathsf{QC*}} \alpha \text{ then } \Gamma \vdash_{\mathsf{QC*}} \beta$$

- Cut

$$\text{If } \Gamma \cup \{\alpha\} \vdash_{\mathsf{QC*}} \beta \text{ and } \Delta \vdash_{\mathsf{QC*}} \alpha \text{ then } \Gamma \cup \Delta \vdash_{\mathsf{QC*}} \beta$$

What's the catch?

- By restricting $\vee$-introduction, do we lose important conclusions?
- Short answer: no.
- See the full paper further discussion.

Conclusion

A simple tweak on quasi-classical logic, results in the logic QC*

- Achieves the properties QC fail at:
  - Cut
  - Deduction
  - Right modus ponens
  - Unit cumulativity
- Preserves the good features of QC
  - Is not explosive.
  - Preserves the usual boolean equivalences.
  - Does not require consistency checking.
  - Supports disjunctive syllogism.
- See http://www.stanford.edu/∼erickao/ for full paper.
- Much simpler proof theory

Philippe Besnard and Anthony Hunter, *Quasi-classical logic: Non-trivializable classical reasoning from incosistent information*, Proceedings of the European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty, 1995, pp. 44–51.

Carl Hewitt, *Common sense for concurrency and strong paraconsistency using unstratified inference and reflection*, CoRR **abs/0812.4852** (2008).

Anthony Hunter, *Reasoning with contradictory information using quasi-classical logic*, Journal of Logic and Computation **10** (1999), 677–703.

Eric J.-Y. Kao, *Two less-than-obvious sources of explosion with respect to quasi-classical logic*, Submitted to Inconsistency Robustness 2011.

What's the catch?

By restricting $\vee$-introduction, do we lose important conclusions?

Example

Consider an airline reservation system.

- f — seat available in first-class
- c — seat available in coach

Suppose the system's knowledge is $\Sigma = \{f\}$

- $\{f\} \vdash_{QC} f \vee c$ (Yes)
- $\{f\} \nvdash_{QC*} f \vee c$ (Don't know)

Fix the UI not the logic

- Q "Is there an available seat in first-class or coach class"
  A2 "I don't know"
  A2 "There is an available seat in first-class"
- Whenever $\Sigma \vdash_{\mathsf{QC}} \phi$, there is a sentence ψ such that $\Sigma \vdash_{\mathsf{QC*}} \psi$ and ψ is stronger than ϕ.[4]

[4] By ϕ is stronger than ψ, we mean that for each clause $c \in \mathrm{Clauses}(\phi)$, there is a clause $d \in \mathrm{Clauses}(\psi)$ that subsumes c.

Information hiding

However, the idea of answering something stronger is not desirable in all situations. Sometimes a weak answer is needed for reason of privacy and security. Consider the following example.

Example

Consider a university knowledge base that contains people information.

- v — student has left a program voluntarily
- n — student has left a program involuntarily

To protect privacy,

- Reveal: whether $v \vee n$
- Hide: whether v
- Hide: whether n

Fix the usage, not the logic

- new proposition l — student has left a program (voluntarily or involuntarily)
- view definition: $\Lambda = \{n \rightarrow l, v \rightarrow l\}$.

As desired[5],

- $\{v\} \cup \Lambda \vdash_{QC*} l$
- $\{n\} \cup \Lambda \vdash_{QC*} l$

[5] This use of view definitions is a standard technique for enforcing security and privacy in databases.

Failed properties

QC fails at several well-known properties of classical logic examined in [Hun99].
For $\Gamma, \Delta \in \mathcal{P}(\mathcal{L}), \alpha, \beta \in \mathcal{L}$,

- Cut

$$\text{If } \Gamma \cup \{\alpha\} \mid\!\sim \beta \text{ and } \Delta \mid\!\sim \alpha \text{ then } \Gamma \cup \Delta \mid\!\sim \beta$$

E.G. $\{\neg p\} \cup \{p \vee q\} \vdash_{QC} q$ and $\{p\} \vdash_{QC} p \vee q$, but $\{\neg p\} \cup \{p\} \nvdash_{QC} q$

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E.G. $\{\neg p\} \vdash_{QC} p \rightarrow q$, but $\{\neg p\} \cup \{p\} \nvdash_{QC} q$.

Failed properties

- Right Modus Ponens (RMP)

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$$\text{If } \Gamma \cup \{\alpha\} \mid\!\sim \beta \text{ and } \Gamma \mid\!\sim \alpha \text{ then } \Gamma \mid\!\sim \beta$$

E.G. $\{\neg p, p\} \vdash_{QC} p \vee q$ and $\{\neg p, p\} \cup \{p \vee q\} \vdash_{QC} q$, but $\{\neg p, p\} \nvdash_{QC} q$

Property	$\vdash_{QC}$	$\vdash_{QC*}$
Reflexivity	Succeeds	Succeeds
Monotonicity	Succeeds	Succeeds
And	Succeeds	Succeeds
Or	Succeeds	Succeeds
Consistency preservation	Succeeds	Succeeds
Subclassicality	Succeeds	Succeeds
Cut	Fails	Succeeds
Deduction	Fails	Succeeds
Right modus ponens	Fails	Succeeds
Unit cumulatively	Fails	Succeeds
Conditionalization	Fails	Fails
Right weakening	Fails	Fails
Left logical equivalence	Fails	Fails
Supraclassicality	Fails	Fails