Powermonads and Tensors of Unranked Effects

Sergey Goncharov and Lutz Schröder

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Monads and Effects: Tools

Strong monad $\mathbb{T}$: Underlying category $\mathcal{C}$, endofunctor $T : \mathcal{C} \to \mathcal{C}$, unit: $\eta : Id \to T$, multiplication $\mu : T^2 \to T$,

plus strength: $\tau_{A,B} : A \times TB \to T(A \times B)$.

Metalanguage of effects:

- $Type_W ::= W \mid 1 \mid Type_W \times Type_W \mid T(Type_W)$
- Term construction (Cartesian operators omitted):

$$\frac{x : A \in \Gamma}{\Gamma \triangleright x : A} \qquad \frac{\Gamma \triangleright t : A}{\Gamma \triangleright f(t) : B}\ (f : A \to B \in \Sigma)$$

$$\frac{\Gamma \triangleright t : A}{\Gamma \triangleright \mathsf{ret}\, t : TA} \qquad \frac{\Gamma \triangleright p : TA \quad \Gamma, x : A \triangleright q : TB}{\Gamma \triangleright \mathsf{do}\ x \leftarrow p; q : TB}$$

Monads and Effects: Usage

Rough idea:

- function spaces are morphisms: $[\![A \rightharpoonup B]\!] = [\![A]\!] \to T[\![B]\!]$;
- sequencing is binding: $[\![x := p; q]\!] = \mathrm{do}\ x \leftarrow [\![p]\!]; [\![q]\!]$;
- values are pure computations: $[\![\mathbf{c}]\!] = \mathrm{ret}[\![\mathbf{c}]\!]$.

Examples:

- Exeptions: $TA = A + E$.
- States: $TA = S \to (S \times A)$.
- Nondeterminism: $TA = \mathcal{P}(A)$, $\mathcal{P}_\omega(A)$, $\mathcal{P}^\star(A)$, ...
- Input/Output: $TA = \mu X.(A + (I \to O \times X))$.
- Continuations: $TA = (X \to R) \to R$.

E.g. for $TX = S \to \mathcal{P}(S \times X)$: $[\![A \rightharpoonup B]\!] \subseteq ([\![A]\!] \times S) \times ([\![B]\!] \times S)$.

This work: Some motivation

- Can one assume 'w.l.o.g.' presence of higher-order term constructors?

  Yes! [Moggi, 1995]: $\mathbf{C} \mapsto \mathbf{Set}^{\mathbf{C}^{op}}$, $T \mapsto \mathrm{Lan}_Y YT$
  ($Y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ is the Yoneda embedding).

- Can one assume presence of nondeterminism so that e.g.:

$$\begin{aligned} \texttt{if } b \texttt{ then } p \texttt{ else } q &:= b?; p + (\neg b)?; q \\ \texttt{while } b \texttt{ do } p &:= (b?; p)^{\star}; (\neg b)? \end{aligned}$$

- How to make formal sense of these assignments (we dub: Fischer-Ladner encoding) w.r.t. generic effects?
- When Fischer-Ladner is conservative?

What about a nondeterminism monad transformer?

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  What about a nondeterminism monad transformer?

Additive monads/Kleene monads

Additive monads: There are deadlock ($\bot$), choice ($+$) which satisfy axioms of semilattices, left/right distribute over binding.
Example: $\mathcal{P}$, $\mathcal{P}_\omega$; non-example: $\mathcal{P}(\_ + E)$, because:

$$\mathrm{do}\ \ x \leftarrow raise(e); \bot = raise(e) \neq \bot.$$

Completely additive: same but w.r.t. complete semilattices.
Example: $\mathcal{P}$; non-example: $\mathcal{P}_\omega$.

Weakly (completely) additive: same minus deadlock.
Examples: same plus $\mathcal{P}(\_ + E)$, $\mathcal{P}^\star$ (non-empty powerset).

(Weak) Kleene monads: (weakly) additive plus Kleene star: $(\mathrm{init}\ \_ \leftarrow \_\ \mathrm{in}\ \_^*)$.

Theorem. Every (weakly) completely additive monad is a (weak) Kleene monad.

Fischer-Ladner encoding

Let $\mathbf{C}$ be a distributive category (products, coproducts plus distributivity); let T be a Kleene monad over $\mathbf{C}$.

Let a test function $? : 2 \to \mathsf{T}1$ be a lattice homomorphism:

$$\top? = \mathsf{ret}\,\star \qquad \bot? = \bot \qquad (\phi \vee \psi)? = \phi? + \psi?$$
$$(\phi \wedge \psi)? = \mathsf{do}\ \phi?; \psi? + \mathsf{do}\ \psi?; \phi?$$

Then put:

$$\begin{aligned} if(b, p, q) &:= \mathsf{do}\ b?; p + \mathsf{do}\ (\neg b)?; q, \\ while(b, p) &:= \mathsf{do}\ x \leftarrow (\mathsf{init}\, x \leftarrow \mathsf{ret}\, x\, \mathsf{in} (\mathsf{do}\ b?; p)^*); \\ &\quad\ \mathsf{do}\ (\neg b)?; \mathsf{ret}\, x \end{aligned}$$

Theorem. If 'if' and 'while' are initially defined in terms of coproducts they do always decompose as above.

Algebraic effects

(Finitary) Lawvere theory: small Cartesian category L plus a strict-product-preserving, identity-on-objects functor: $I : \mathbb{N}^{op} \to L$ ($\mathbb{N}$ = naturals and maps with summs as coproducts.)

- $L(n, 1)$ — operations; $L(0, 1)$ — constants;
- $\mathrm{Mod}(L, C) \subseteq \mathrm{Fun}(L, C)$ — models of L in C;
- forgetful functor $\mathrm{Mod}(L, C) \to C$ leads to finitary monads.

Finite nondeterminism: one constant $\perp : 0 \to 1$, one operation: $+ : 2 \to 1$. Then e.g. $(\lambda a, b, c.\, a + b + c) : 3 \to 1$, $(\lambda a.\, \langle a, \perp\rangle) : 1 \to 2$, etc.

States: $\mathrm{lookup}_l : V \to 1$, $\mathrm{update}_{l,v} 1 \to 1$ ($l \in L$, $v \in V$).
E.g.: $\mathrm{update}_{l,v}(\mathrm{lookup}_l \langle p_1, \ldots, p_{|V|}\rangle) = \mathrm{update}_{l,v}(p_v)$.

Large Lawvere theory: L has all small products;
$I : \mathbf{Set}^{op} \to L$ is strict-small-product-preserving, id-on-objects.

Theorem [Linton, 1966]: Large Lawvere theories = Monads on $\mathbf{Set}$.

Sum and Tensor

Sum of effects: blind union of signatures,
e.g. $\Sigma^\star + T = \mu\gamma.T(\Sigma\gamma + \_)$ ($\Sigma^\star$ = I/O, Resumptions, Exeptions.)

Tensor = Sum modulo commutativity of operations:

$$\begin{tikzcd}
n_1 \times n_2 \arrow[r, "n_1 \otimes f_2"] \arrow[d, "f_1 \otimes n_2"'] & n_1 \times m_2 \arrow[d, "f_1 \otimes m_2"] \\
m_1 \times n_2 \arrow[r, "m_1 \otimes f_2"] & m_1 \times m_2.
\end{tikzcd}$$

($n \otimes f = f \times \ldots \times f$ 'n times'.)

For instance: $\mathrm{lookup}_l\langle p_1 + q_1, p_2 + q_2\rangle$
$= \mathrm{lookup}_l\langle p_1, p_2\rangle + \mathrm{lookup}_l\langle q_1, q_2\rangle$.

Examples: $(\_ \times S)^S \otimes T = T(\_ \times S)^S$, $(\_)^S \otimes T = T^S$,
$(M \times \_) \otimes T = T(M \times \_)$ where M is a monoid (of messages).

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Nondeterminism transformer, aka Powermonad

Given a monad T over $\mathbf{Set}$ we can (possibly) form $\mathsf{T} \otimes \mathcal{P}$.

If $\mathsf{T} \otimes \mathcal{P}$ exists then it is completely additive.

Theorem. For the forgetful functor $\mathcal{U}$ from completely additive monads to monads the following is equivalent:

- $\mathcal{U}$ has a left adjoint;
- $\mathcal{U}$ is monadic;
- $\mathsf{T} \otimes \mathcal{P}$ exists and $\mathsf{T} \mapsto \mathsf{T} \otimes \mathcal{P}$ lifts to a left adjoint of $\mathcal{U}$.

Analogously for $\mathcal{P}^{\star}$, $\mathcal{P}_{\omega}$, $\mathcal{P}^{\star}_{\omega}$, etc.

Q: But when do the appropriate tensors exist?
A: Well, general existence of tensors it open from 1969 [Manes, 1969].

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Analogously for $\mathcal{P}^\star$, $\mathcal{P}_\omega$, $\mathcal{P}_\omega^\star$, etc.

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Uniformity (1/2)

From [Hyland, Plotkin, and Power, 2003], [Hyland, Levy, Plotkin, and Power, 2007] we know:

- tensors of ranked ($\approx$ algebraic) monads always exist;
- tensors of ranked monads with continuations exist;
- tensors with states always exist.

Not covered: $\mathcal{P} \otimes T$ ($\mathcal{P}$ is unranked!), $\mathcal{P}_\omega \otimes T$ unless T ranked.

More unranked monads: continuations, selections [Escardó and Oliva, 2010], families ($\mathcal{P}\mathcal{P}$).

Idea of uniformity: for countable powersets Lawvere operation are 'instances' of a big union, e.g.: $a \cup b = a \cup b \cup b \cup \cdots$.
Analogously for $\mathcal{P}$, but big union is 'unobservable'.

Informal definition: for every family of operations $\{f_i : n \to 1\}_i$ there is $\hat{f} : k \to 1$ such that $f_i = \hat{f}\sigma_i$;
Here: σ_i (i)rearranges arguments; (ii) plugs in constants.

Uniformity (2/2)

Non-existences of tensors is a cardinality issue.

Theorem. Tensor of two monads over $\mathbf{Set}$ does exist, provided one of them is uniform.

Idea of the proof: resorting composite operations of the tensor using uniformity. Like this:

$$f(a \cup b, c) \to f(a \cup b, c \cup c) \to f(a, c) \cup f(b, c).$$

In general, terms of $T_1 \otimes T_2$ with uniform T_1 have three layers: (i) operations of T_1, (ii) operations of T_2, (iii) constants of T_1.

Uniform monads: $\mathcal{P}$, $\mathcal{P}^\star$, $\mathcal{P}_{\omega 1}$, $\mathcal{P}^\star_{\omega 1}$ (but not $\mathcal{P}_\omega$!), infinite multisets and (surprisingly!) continuations T_R.

Quick intuition: $L_{T_R}(n, 1) \simeq R^n \to R$. Hence for $|R| > 1$, given $f_1, f_2 : R^n \to R$, we can unify them into $\hat{f} : R^{n+1} \to R$:

$$\hat{f}(c) := \mathit{if}\ (c(\mathsf{inr}\,\star) = r_0)\ \mathit{then}\ f_1(c \circ \mathsf{inl})\ \mathit{else}\ f_2(c \circ \mathsf{inl})$$

Conservativity

When $T \mapsto T \otimes \mathcal{P}$, $T \mapsto T \otimes \mathcal{P}^\star$ are conservative?

For $\mathcal{P}$ ($\mathcal{P}^\star$ is analogous): $L_{T \otimes \mathcal{P}}(n, m)$ is $\mathcal{P}(L_T(n, m))$ modulo rectangular equivalence, smallest equivalence closed under:

$$L_T(0, n) \approx \emptyset \qquad\qquad \frac{\forall i.\, \pi_i A \approx \pi_i B}{CA \approx CB}$$

Example: $\{\langle a, b\rangle, \langle c, d\rangle\} \approx \{\langle a, d\rangle, \langle c, b\rangle\}$.

Corollary: Conservativity fails unless $|T\emptyset| = 0$ or $|T\emptyset| = 1$

Definition: a monad T is bounded if $T\emptyset = \{\bot\}$.

Examples include besides $\mathcal{P}$, partial states, nondeterministic states with deadlock, output with nontermination: $O^\star \times \_ + 1$.

Bounded monads

Approximation

Smallest preorder with $\perp$ as the bottom and closed under

$$\frac{\forall i.\, \pi_i f \sqsubseteq \pi_i g}{hf \sqsubseteq hg}$$

This instantiates as expected, e.g. for lists: $[a, c] = [a, \perp, c] \sqsubseteq [a, b, c]$.

Closure

For $A \subseteq L(n, m)$, $\mathsf{cl}(A)$ is the closure of A under approximations and under

$$\frac{\forall i.\; g\Delta_i h \in \mathsf{cl}(A)}{hg \in \mathsf{cl}(A)}$$

Here: $\Delta_i = \prod_j \delta_{ij} \in L(n, n)$,

$\delta_{ii} = id$, $\delta_{ij} = \perp$ for $i \neq j$.

Theorem. If for all $f \in L_T(n, m)$, $\mathsf{cl}(f) = \{g \mid g \sqsubseteq f\}$ then $T \mapsto T \otimes \mathcal{P}$ is injective iff $\sqsubseteq$ is a partial order.

But does the tensor always exist?!

NO!

Well-order monad [Goncharov and Schröder, 2011]:

$$\mathcal{W}X = \{(Y, \rho) \mid Y \subseteq X, \rho \text{ a well-order on } Y\}.$$

equivalently: non-repetitive strict ordinal-indexed lists ($a + a = \bot$, $a + \bot = \bot$).

Theorem. Tensor product of $\mathcal{W}$ with the free monad of two binary operations does not exist.

Corollary. Tensor product with the finite list monad does not always exist.

Open question: tensoring with the finitary powerset.

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The End

Thanks for your attention!

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