Decidability of Definability

Manuel Bodirsky

CNRS / LIX, École Polytechnique

Joint work with Michael Pinsker and Todor Tsankov

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2. Existential definability
3. Existential positive definability
4. Primitive positive definability ($\forall, \vee, \neg$ forbidden): formulas of the form

$$\exists x_1, \ldots, x_n. \psi_1 \wedge \cdots \wedge \psi_m$$

where $\psi_1, \ldots, \psi_m$ are atomic τ-formulas

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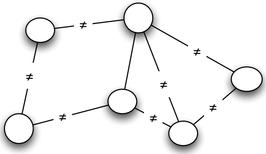

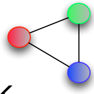

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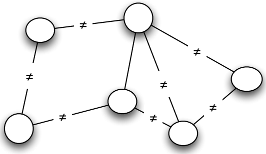

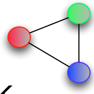

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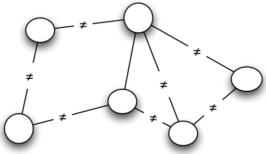

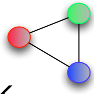

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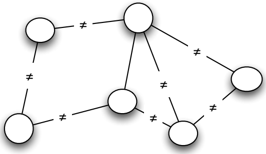

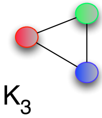

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Infinite Γ: many additional examples in artificial intelligence, computer algebra, computational linguistics, computational biology, scheduling, ...

Primitive Positive Definability

Lemma (Jeavons et al.,1997).

Let $\Gamma = (D; R_1, \ldots, R_k)$ be a relational structure, and
let R be a relation that has a primitive positive definition in Γ.
Then CSP(Γ) and CSP$(D; R, R_1, \ldots, R_k)$ are polynomial-time equivalent.

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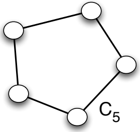

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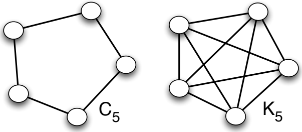

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Lemma (Jeavons et al.,1997).

Let $\Gamma = (D; R_1, \ldots, R_k)$ be a relational structure, and
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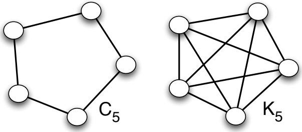

Example. Claim: $\mathrm{CSP}(C_5)$ is NP-hard.
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$$E'(x,y) \equiv \exists p_1, p_2, p_3, q_1, q_2. \big(E(x,p_1) \wedge E(p_1,p_2) \wedge E(p_2,p_3) \wedge E(p_3,y) \\ \wedge E(x,q_1) \wedge E(q_1,q_2) \wedge E(q_2,y)\big)$$

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- (Schaefer'78) Let Γ be a structure with domain $\{0, 1\}$.
  Then CSP(Γ) is in P, or the following relation is primitive positive definable in Γ, and CSP(Γ) is NP-complete:

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- (B.+Pinsker'11) Let Γ be first-order definable in the Rado graph.
  Then $\text{CSP}(\Gamma)$ is in P, or one out of four relations is primitive positive definable in Γ and $\text{CSP}(\Gamma)$ is NP-complete.

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Observation: If $\mathrm{Expr}(\Gamma)$ is decidable, then $\mathrm{Expr}(\Delta)$ is decidable for all structures Δ that are definable in Γ.

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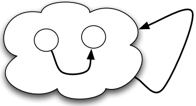

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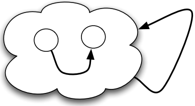

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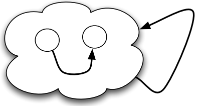

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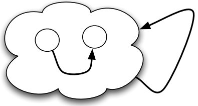

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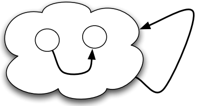

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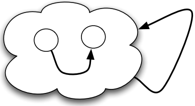

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An ω-categorical Γ is homogeneous if and only if Γ has quantifier elimination

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Curious phenomenon:

Observation (B.,Foniok,Nešetřil,Pinsker,Todorcevic,Tsankov)

All known homogeneous structures Γ can be expanded to a homogeneous Ramsey structure.

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Theorem 4 (Nešetřil+Rödl'83).

All Henson digraphs can be expanded to homogeneous Ramsey structures.

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Want to prove:

Theorem.

Let Γ be homogeneous, Ramsey, and finitely bounded.
Then Expr(Γ) is decidable.

Tool: Preservation Theorems

A function $f : D^k \to D$ preserves $R \subseteq D^m$ if $\big(f(a_1^1, \dots, a_1^k), \dots, f(a_m^1, \dots, a_m^k)\big) \in R$ whenever $(a_1^i, \dots, a_m^i) \in R$ for all $i \leq m$.

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In finite or ω-categorical structures Γ, a relation R is

- primitively positively definable if and only if R is preserved by all polymorphisms of Γ (MB+Nešetřil'03);

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- existentially positively definable in Γ if and only if R is preserved by all endomorphisms of Γ (Lyndon and Los-Tarski combined);

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- primitively positively definable if and only if R is preserved by all polymorphisms of Γ (MB+Nešetřil'03);
- existentially positively definable in Γ if and only if R is preserved by all endomorphisms of Γ (Lyndon and Los-Tarski combined);
- existentially definable in Γ if and only if R is preserved by all self-embeddings of Γ into Γ (Los-Tarski);

Tool: Preservation Theorems

A function $f : D^k \to D$ preserves $R \subseteq D^m$ if $\big(f(a_1^1, \ldots, a_1^k), \ldots, f(a_m^1, \ldots, a_m^k)\big) \in R$ whenever $(a_1^i, \ldots, a_m^i) \in R$ for all $i \leq m$.

We say that f is a polymorphism of Γ if it preserves all relations of Γ.

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- existentially definable in Γ if and only if R is preserved by all self-embeddings of Γ into Γ (Los-Tarski);
- first-order definable in Γ if and only if R is preserved by all automorphisms of Γ (Ryll-Nardzewski).

Canonical Mappings

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Definition (Canonical Unary Operations)

An operation $f : \Gamma \rightarrow \Delta$ is canonical if for all k-types t_1 in Γ there exists a k-type t_2 in Δ such that f maps every tuple of type t_1 to a tuple of type t_2.

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Can be generalized to higher-ary operations

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For this we need:
if Γ is Ramsey, and $a_1, \ldots, a_k$ are elements of Γ,
then $(\Gamma, a_1, \ldots, a_k)$ is Ramsey as well.

Topological Dynamics

Problem: If Γ is Ramsey, is $(\Gamma, c_1, \dots, c_n)$ also Ramsey?

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If Γ is ordered homogeneous Ramsey, then so is $(\Gamma, c_1, \ldots, c_n)$.

Fact 2:

Every homogeneous Ramsey structure Γ can be expanded to an ordered homogeneous Ramsey structure (Todorcevic, Van Thé).

Concluding Remarks, Open Problems

Contributions:

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