Large Deviations 3

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Toronto, April 15, 2011

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- arxiv.org/pdf/1008.1946

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- We are interested in the behavior of large random graphs. $\mathcal{G}_n$
- $\mathcal{G}_n$ is a graph with n vertices.
- Each of the $\binom{n}{2}$ edges are turned on independently with probability p.
- We then have a probability measure Q_n on the space of $2^{\binom{n}{2}}$ possible graphs.

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$$\frac{N}{n^3} \simeq \frac{p^3}{6}$$

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$$\frac{N}{n^3} \simeq a$$

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- What about the probability for two such rare events happening together?
- What kinds of graphs contribute to these events?

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- First Step. Need a space independent of n in which graphs of all sizes live.
- Imbedding the vertices $\{1, 2, \ldots, n\}$ as subintervals of the unit interval $[0, 1]$
- $E_i^n = [\frac{i-1}{n}, \frac{i}{n}]$
- The the graph V imbedded as a random function

$$f(x, y) = \sum_{i,j} \mathbf{1}_{E_i^n}(x)\mathbf{1}_{E_j^n}(y)\pi_{i,j}$$

$\pi_{i,j} = 1$ if the edge (i, j) is present and 0 otherwise.

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- But the topology is too weak.
- Triangle Count $\Delta(f)$

$$\frac{1}{6}\int_0^1\int_0^1\int_0^1 f(x_1,x_2)f(x_2,x_3)f(x_3,x_1)dx_1dx_2dx_3$$

is not continuous.

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- $\{f(x, y)\}$ live only on functions that are either 0 or 1
- It is hard to make them converge to a more general function like a constant p

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- Weak topology

$$\int f_n(x,y)\phi(x,y)dxdy \to \int f(x,y)\phi(x,y)dxdy$$

for every ϕ bounded by 1.

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- Strong Topology

$$\sup_{|\phi|\leq 1} |\int [f_n(x,y) - f(x,y)]\phi(x,y)dxdy| \to 0$$

- Cut Topology.

$$\sup_{\substack{|\phi|\le 1\\ |\psi|\le 1}} \left| \int [f_n(x,y) - f(x,y)]\phi(x)\psi(y)dxdy \right| \to 0$$

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- A little weaker than strong topology.
- Let γ be a finite graph, with vertices $1,2,\ldots,k$ and some edges (un-oriented) $e \in E$.
- The functional

$$\Phi_\gamma(f) = \int \Pi_{e\in E} f(x_i,x_j)dx_1dx_2\ldots dx_k$$

is a continuos functional of f in the cut topology.

Proof

- Consider triangles.
  $\int f_n(x_1, x_2) f_n(x_2, x_3) f_n(x_3, x_1) dx_1 dx_2 dx_3$

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  $\to 0$ not only for each x_3 but uniformly over x_3.
- Therefore
  $$\int (f_n - f)(x_1, x_2) f_n(x_2, x_3) f_n(x_3, x_1) dx_1 dx_2 dx_3$$
  $\to 0$

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- Replace f_n by f, one edge at a time.
- It works.
- $$\int [f_n - f](x_i, x_j)\phi_n(x_i, x^*)\psi_n(x_j, x^*)dx_i dx_j dx^*$$
- Goes to 0.

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- $$d_{\square}(f,g) = \sup_{\substack{|\phi|\leq 1 \\ |\psi|\leq 1}} \left| \int [f(x,y) - g(x,y)]\phi(x)\psi(y)\,dx\,dy \right|$$

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$$d_{\square}(f,g) = \sup_{A,B} \left| \int_{x\in A, y\in B} [f(x,y) - g(x,y)]dxdy \right|$$

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- Is the law of large numbers valid in the cut topology?

- $$\sup_{A,B}\left|\frac{N(A,B)}{n^2}-|A||B|\right|\to 0?$$

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- Sup is the problem.
- Enough to take A, B to be union of intervals $[\frac{i-1}{n}, \frac{i}{n}]$.

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- $2^n \times 2^n << e^{cn^2}$

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- $$P(A) = \int_A e^{-\log \frac{dQ}{dP}} dQ$$
- $$= Q(A)[\frac{1}{Q(A)} \int_A e^{-\log \frac{dQ}{dP}} dQ]$$
- $$\geq Q(A) \exp[-[\frac{1}{Q(A)} \int_A \log \frac{dQ}{dP} dQ]]$$

- $$P_n[F \in B(f,\delta)] \geq \exp[-\frac{n^2}{2}\int H_p(f(x,y))dxdy + o(n^2)]$$

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- Where

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- There is no chance of coercivity unless $\mathcal{X}$ is compact.

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- $\int f_n(x,y)dy \to \int f(x,y)dy$ in $L_1[0,1]$
- Kills any chance for $\mathcal{X}$ being compact.

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- Symmetry with respect to the group G of measure preserving one to one maps σ of $[0,1] \to [0,1]$.

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- Orbits $\tilde{f} = \{f(\sigma x, \sigma y)\}$
- $$\begin{aligned} d_\square(\tilde{f}, \tilde{g}) &= \inf_{\sigma_1, \sigma_2} d_\square(f_{\sigma_1}, g_{\sigma_2}) \\ &= \inf_{\sigma} d_\square(f_\sigma, g) \\ &= \inf_{\sigma} d_\square(f, g_\sigma) \end{aligned}$$

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- This is a consequence of Szemerédi's regularity lemma.

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- $$d_\square(f, \sum_{i,j} \mathbf{1}_{E_i^n}(x)\mathbf{1}_{E_j^n}(y)\pi_{i,j}) \leq \epsilon$$

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- Euler equation for the function at which the infimum is attained.

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$$\frac{1}{2}[p_c \log \frac{p_c}{p} + (1 - p_c) \log \frac{1 - p_c}{1 - p}]$$
$$> \frac{1}{2}[p_c^2 \log \frac{1}{p} + (1 - p_c^2) \log \frac{1}{1 - p}]$$

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$$\begin{aligned} I(p_c) &= \frac{1}{2}[p_c \log \frac{p_c}{p} + (1-p_c)\log \frac{1-p_c}{1-p}] \\ &\simeq \frac{1}{2}\log(1-p) \end{aligned}$$

- Can fix $\Phi_\gamma(f)$ for any finite number of γ's and minimize $I_p(f)$.

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- Scale down.

$$\sigma_i^n(\omega) = \frac{\lambda_i^n(\omega)}{n}$$

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- Estimates in the scale

$$P(E) \simeq \exp[-cn^2 + o(n^2)]$$

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- $$c = c(\mathcal{S})$$
- What is it?

$$c(\mathcal{S}) = \inf_{\{\phi_j(x)\} \in \mathcal{B}} \frac{1}{2} \int H\Big(\sum_j \sigma_j \phi_j(x) \phi_j(y)\Big) dx dy$$

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- $$\mathcal{K}(\mathcal{S}) = \{k : \sigma(k) = \mathcal{S} \cup \{0\}\}$$

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- $$H(k) = \sup_{\theta} [\theta\, k - \log E[e^{\theta X}]]$$

  is the Cramér rate function.

- $$H(k) \geq c\, k^2$$

  provides compactness.

Proof

- Imbed

$$k(x, y) = \sum_{i,j} X_{i,j} \mathbf{1}_{[\frac{i-1}{n}, \frac{i}{n}]}(x) \mathbf{1}_{[\frac{j-1}{n}, \frac{j}{n}]}(y)$$

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Proof

- Imbed

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- Do LDP on $\mathcal{K}$ in the cut-topology
- $\mathcal{S}(k)$ depends continuously on k.

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- $\to k$ in cut-topology.

last slide

THANK YOU

last slide

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  THE END