

Large Deviations 3

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Toronto, April 15, 2011

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 - arxiv.org/pdf/1008.1946

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- Large deviations of Erdős-Renyi Random graphs.
- We are interested in the behavior of large random graphs. \mathcal{G}_n
- \mathcal{G}_n is a graph with n vertices.
- Each of the $\binom{n}{2}$ edges are turned on independently with probability p .
- We then have a probability measure Q_n on the space of $2^{\binom{n}{2}}$ possible graphs.

- Random symmetric $n \times n$ matrix of 0's and 1's.

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$$\frac{N}{n^3} \simeq \frac{p^3}{6}$$

- What is the probability that

$$\frac{N}{n^3} \simeq a$$

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- What about the probability for two such rare events happening together?
- What kinds of graphs contribute to these events?

- First Step. Need a space independent of n in which graphs of all sizes live.

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- First Step. Need a space independent of n in which graphs of all sizes live.
- Imbedding the vertices $\{1, 2, \dots, n\}$ as subintervals of the unit interval $[0, 1]$
- $E_i^n = [\frac{i-1}{n}, \frac{i}{n}]$
- The the graph V imbedded as a random function

$$f(x, y) = \sum_{i,j} \mathbf{1}_{E_i^n}(x) \mathbf{1}_{E_j^n}(y) \pi_{i,j}$$

$\pi_{i,j} = 1$ if the edge (i, j) is present and 0 otherwise.

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- But the topology is too weak.
- Triangle Count $\Delta(f)$

$$\frac{1}{6} \int_0^1 \int_0^1 \int_0^1 f(x_1, x_2) f(x_2, x_3) f(x_3, x_1) dx_1 dx_2 dx_3$$

is not continuous.

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- $\{f(x, y)\}$ live only on functions that are either 0 or 1
- It is hard to make them converge to a more general function like a constant p

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- Weak topology

$$\int f_n(x, y) \phi(x, y) dx dy \rightarrow \int f(x, y) \phi(x, y) dx dy$$

for every ϕ bounded by 1.

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for every ϕ bounded by 1.

- Strong Topology

$$\sup_{|\phi| \leq 1} \left| \int [f_n(x, y) - f(x, y)] \phi(x, y) dx dy \right| \rightarrow 0$$

■ Cut Topology.

$$\sup_{\substack{|\phi| \leq 1 \\ |\psi| \leq 1}} \left| \int [f_n(x, y) - f(x, y)] \phi(x) \psi(y) dx dy \right| \rightarrow 0$$

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- Let γ be a finite graph, with vertices $1, 2, \dots, k$ and some edges (un-oriented) $e \in E$.

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- A little weaker than strong topology.
- Let γ be a finite graph, with vertices $1, 2, \dots, k$ and some edges (un-oriented) $e \in E$.
- The functional

$$\Phi_\gamma(f) = \int \prod_{e \in E} f(x_i, x_j) dx_1 dx_2 \dots dx_k$$

is a continuous functional of f in the cut topology.

Proof

- Consider triangles.

$$\int f_n(x_1, x_2) f_n(x_2, x_3) f_n(x_3, x_1) dx_1 dx_2 dx_3$$

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→ 0 not only for each x_3 but uniformly over x_3 .

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→ 0 not only for each x_3 but uniformly over x_3 .

- Therefore

$$\int (f_n - f)(x_1, x_2) f_n(x_2, x_3) f_n(x_3, x_1) dx_1 dx_2 dx_3$$

→ 0

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$$\int [f_n - f](x_i, x_j) \phi_n(x_i, x^*) \psi_n(x_j, x^*) dx_i dx_j dx^*$$

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- It works.

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$$\int [f_n - f](x_i, x_j) \phi_n(x_i, x^*) \psi_n(x_j, x^*) dx_i dx_j dx^*$$

- Goes to 0.

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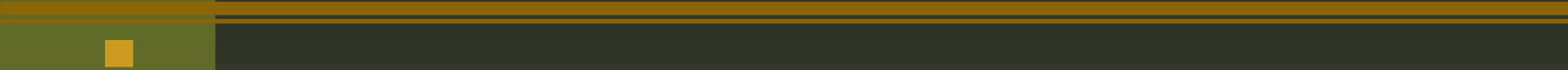



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- Is the law of large numbers valid in the cut topology?


$$\sup_{A,B} \left| \frac{N(A, B)}{n^2} - |A||B| \right| \rightarrow 0?$$



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■

$$\sup_{A,B} \left| \frac{N(A, B)}{n^2} - |A||B| \right| \rightarrow 0?$$

- Each one is just law of large numbers.
- Sup is the problem.
- Enough to take A, B to be union of intervals $\left[\frac{i-1}{n}, \frac{i}{n}\right]$.

- There are $2^n \times 2^n$ of them.

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- The LLN comes with an error estimate of e^{-cn^2} .
- $2^n \times 2^n \ll e^{cn^2}$

■ Large Deviation Lower Bounds:

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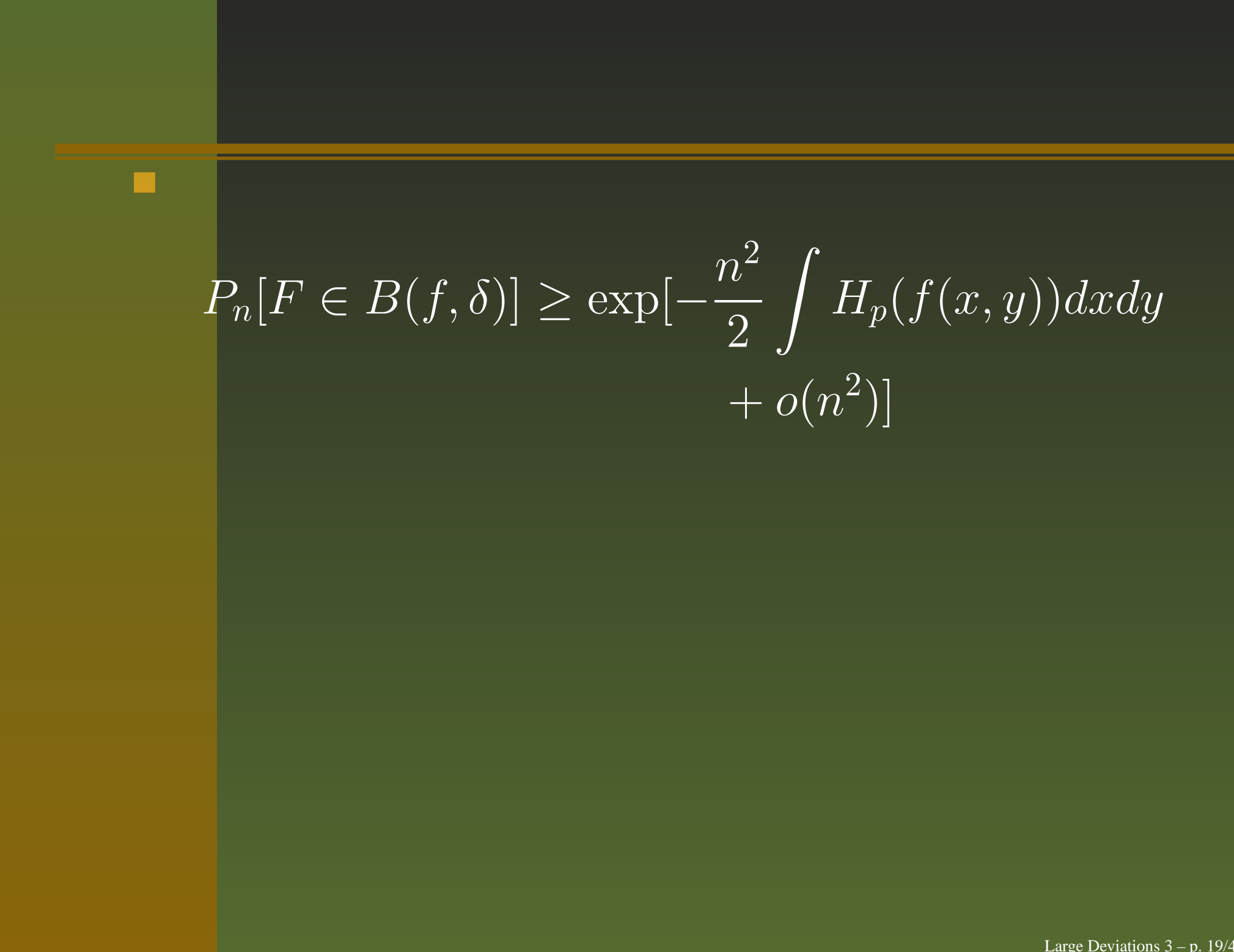
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-

$$\geq Q(A) \exp \left[- \left[\frac{1}{Q(A)} \int_A \log \frac{dQ}{dP} dQ \right] \right]$$

A decorative vertical bar on the left side of the slide, transitioning from dark green at the top to a lighter olive green at the bottom. A thin horizontal line in a golden-brown color crosses the slide, intersecting the vertical bar.
$$P_n[F \in B(f, \delta)] \geq \exp\left[-\frac{n^2}{2} \int H_p(f(x, y)) dx dy + o(n^2)\right]$$



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■ Where

$$H_p(f) = \left[f \log \frac{f}{p} + (1 - f) \log \frac{1 - f}{1 - p}\right]$$

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$$I_p(f) = \frac{1}{2} \int H_p(f(x, y)) dx dy$$

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- There is no chance of coercivity unless \mathcal{X} is compact.

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- $d_{\square}(f_n(x, y), f(x, y)) \rightarrow 0$ implies
- $\int f_n(x, y)dy \rightarrow \int f(x, y)dy$ in $L_1[0, 1]$
- Kills any chance for \mathcal{X} being compact.

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- Symmetry with respect to the group G of measure preserving one to one maps σ of $[0, 1] \rightarrow [0, 1]$.

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$$\begin{aligned} d_{\square}(\tilde{f}, \tilde{g}) &= \inf_{\sigma_1, \sigma_2} d_{\square}(f_{\sigma_1}, g_{\sigma_2}) \\ &= \inf_{\sigma} d_{\square}(f_{\sigma}, g) \\ &= \inf_{\sigma} d_{\square}(f, g_{\sigma}) \end{aligned}$$

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- This is a consequence of Szemerédi's regularity lemma.

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$$f = \sum_{i,j=1}^m \mathbf{1}_{E_i^m}(x) \mathbf{1}_{E_j^m}(y) \pi_{i,j} \in \mathcal{X}$$

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■

$$d_{\square}(f, \sum_{i,j} \mathbf{1}_{E_i^n}(x) \mathbf{1}_{E_j^n}(y) \pi_{i,j}) \leq \epsilon$$

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- The infimum is attained
- Euler equation for the function at which the infimum is attained.

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$$\begin{aligned} & \frac{1}{2} \left[p_c \log \frac{p_c}{p} + (1 - p_c) \log \frac{1 - p_c}{1 - p} \right] \\ & > \frac{1}{2} \left[p_c^2 \log \frac{1}{p} + (1 - p_c^2) \log \frac{1}{1 - p} \right] \end{aligned}$$

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$$\begin{aligned} I(p_c) &= \frac{1}{2} \left[p_c \log \frac{p_c}{p} + (1 - p_c) \log \frac{1 - p_c}{1 - p} \right] \\ &\simeq \frac{1}{2} \log(1 - p) \end{aligned}$$

- Can fix $\Phi_\gamma(f)$ for any finite number of γ 's and minimize $I_p(f)$.

Random Matrices



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- Scale down.

$$\sigma_i^n(\omega) = \frac{\lambda_i^n(\omega)}{n}$$

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- Estimates in the scale

$$P(E) \simeq \exp[-cn^2 + o(n^2)]$$

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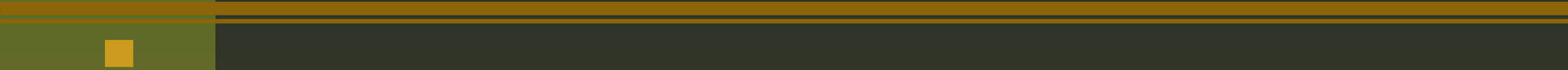
$$c = c(\mathcal{S})$$


- Only possibility is $\mathcal{S} = \{\sigma_j\}$ such that $|\sigma_j| \rightarrow 0$.



$$c = c(\mathcal{S})$$

- What is it?


$$c(\mathcal{S}) = \inf_{\{\phi_j(x)\} \in \mathcal{B}} \frac{1}{2} \int H\left(\sum_j \sigma_j \phi_j(x) \phi_j(y)\right) dx dy$$



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- $\{\phi_j\}$ are orthonormal. \mathcal{B} all choices of orthonormal sets.

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$$\mathcal{K}(\mathcal{S}) = \{k : \sigma(k) = \mathcal{S} \cup \{0\}\}$$


$$c(\mathcal{S}) = \inf_{k \in \mathcal{K}} \frac{1}{2} \int H(k(x, y)) dx dy$$

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$$H(k) \geq c k^2$$

provides compactness.

Proof

■ Imbed

$$k(x, y) = \sum_{i,j} X_{i,j} \mathbf{1}_{[\frac{i-1}{n}, \frac{i}{n}]}(x) \mathbf{1}_{[\frac{j-1}{n}, \frac{j}{n}]}(y)$$

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- Do LDP on \mathcal{K} in the cut-topology
- $\mathcal{S}(k)$ depends continuously on k .

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- $\rightarrow k$ in cut-topology.

last slide

THANK YOU

last slide

THANK YOU

THE END

