

Large Deviations

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- Instead we have probabilities for "yes" and "no".

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- Asymptotic methods are needed.

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- How small?

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- Let us look at some examples.

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- At what rate?

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- Is there a recipe for computing $H(p)$?

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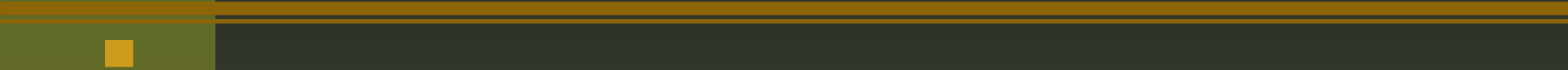
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- Change the model, but do it frugally!

General Formulation

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- $c(A) = \inf_{x \in A} I(x)$

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- If $\inf_{x \in A^\circ} I(x) = \inf_{x \in \bar{A}} I(x)$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log P_n(A) = - \inf_{x \in A} I(x)$$

Some simple facts

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$$\int \exp[nF(x)] dP_n = \exp[n \sup_x [F(x) - I(x)] + o(n)]$$

Ventcell-Freidlin Theory



$$dx(t) = b(x(t))dt, x(0) = x$$

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- Exponential

$$\exp\left[-\frac{1}{\epsilon}I(f(\cdot))\right]$$

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$$\begin{aligned} \frac{dQ_\epsilon}{dP_\epsilon} = & \exp\left[\frac{1}{\epsilon} \int_0^T (f'(t) - b(x(t)))(dx(t) - b(x(t))dt) \right. \\ & \left. - \frac{1}{2\epsilon} \int_0^T \|f'(t) - b(x(t))\|^2 dt \right] \end{aligned}$$

Exit problem



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$$I(\mu) = \sup_{u > 0} - \int \frac{\mathcal{A}u}{u} d\mu$$

Reversible case

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$$\lambda(V) = \sup_{g: \|g\|_{2,\alpha}=1} \left[\int V g^2 d\alpha - \mathcal{D}(g) \right]$$

Reversible case

- Topology of X .

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- $d = 1$, Brownian motion, L_1 or similar.

Future

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- Random graphs, Random matrices

THE END