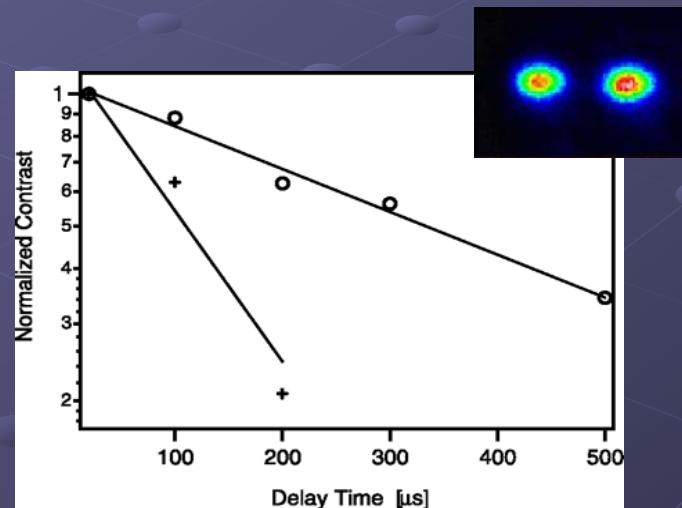


Part 2: Mostly Dynamical Decoupling

Need a way to deal with symmetry breaking...



Dealing with Symmetry Breaking: Creating Collective Dephasing Conditions

L.-A. Wu, D.A.L., *Phys. Rev. Lett.* **88**, 207902 (2002)

General two-qubit dephasing: $H_{SB} = \sigma_1^z \otimes B_1 + \sigma_2^z \otimes B_2$

$$= \underbrace{\frac{1}{2}(\sigma_1^z - \sigma_2^z)}_{Z} \otimes (B_1 - B_2) + \frac{1}{2}(\sigma_1^z + \sigma_2^z) \otimes (B_1 + B_2)$$

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Assume: controllable exchange $X = \frac{1}{2}(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y) + \alpha \sigma_1^z \sigma_2^z$.

$$\{X, Z\} = 0 \quad \Rightarrow \quad XZX = -Z$$

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“Time reversal” Dynamical Decoupling pulse sequence:

$$\exp(-iH_{SB}t) \left[\exp\left(-i\frac{\pi}{2}X\right) \exp(-iH_{SB}t) \exp\left(i\frac{\pi}{2}X\right) \right] = \exp\left(-it\underbrace{(\sigma_1^z + \sigma_2^z) \otimes (B_1 + B_2)}_{\text{Collective Dephasing}}\right)$$



Heisenberg is “Super-Universal”

Same method works, e.g., for *spin-coupled quantum dots QC*:

By BB pulsing of $H_{\text{Heis}} = \frac{J}{2}(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + \sigma_1^z\sigma_2^z)$

collective **decoherence** conditions can be created:

$$H_{\text{SB}} = \sum_{i=1}^n g_i^x \sigma_i^x \otimes B_i^x + g_i^y \sigma_i^y \otimes B_i^y + g_i^z \sigma_i^z \otimes B_i^z \\ \rightarrow S_x \otimes B_x + S_y \otimes B_y + S_z \otimes B_z$$

Requires sequence of 6 $\pi/2$ pulses to create collective decoherence conditions over blocks of 4 qubits. Leakage elimination requires 7 more pulses.

Details: L.-A. Wu, D.A.L., *Phys. Rev. Lett.* **88**, 207902 (2002); L.A. Wu, M.S. Byrd, D.A.L., *Phys. Rev. Lett.* **89**, 127901 (2002).

Earlier DFS work showed universal QC with Heisenberg interaction alone possible [Bacon, Kempe, D.A.L., Whaley, *Phys. Rev. Lett.* **85**, 1758 (2000)]:

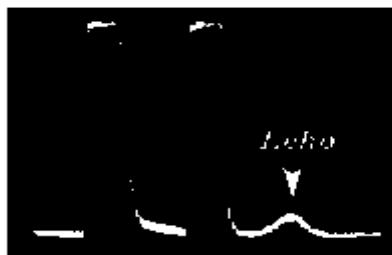
All ingredients available for Heisenberg-only QC

Generalize this idea?
NMR to the Rescue:
Removal of Decoherence via Spin Echo

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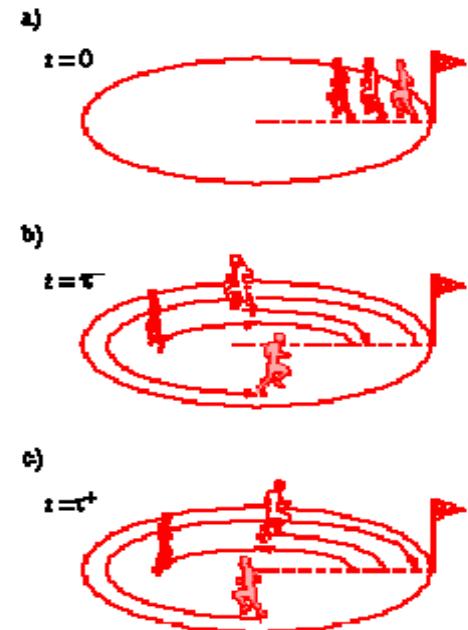
From: Coherent averaging techniques

Coherent control of nuclear spin Hamiltonians in high-resolution NMR spectroscopy.



E.L. Hahn, PR **80**, 580 (1950);
U. Haeberlen & J.S. Waugh, PR **175**, 453 (1968).

Hahn spin echo idea



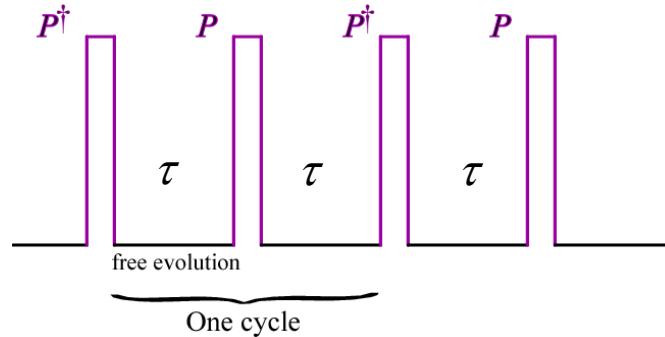
The “race-track” echo:
Effective time reversal

Dynamical Decoupling Basics

A pulse producing a unitary evolution P , such that

$$PH_{\text{SB}}P^\dagger = -H_{\text{SB}} \quad \text{i.e., } \{P, H_{\text{SB}}\} = 0$$

(CPMG, Hahn spin-echo)



Ideal (zero-width) pulses, and ignoring H_B :

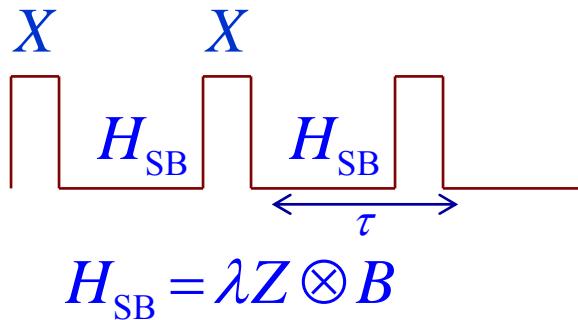
$$\begin{aligned} P \exp(-i\tau H_{\text{SB}}) P^\dagger \exp(-i\tau H_{\text{SB}}) &= \exp(-i\tau PH_{\text{SB}}P^\dagger) \exp(-i\tau H_{\text{SB}}) \\ &= \exp(i\tau H_{\text{SB}}) \exp(-i\tau H_{\text{SB}}) = I \end{aligned}$$

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$XZX = -Z \Rightarrow$
"time reversal",
 H_{SB} averaged to zero
(in 1st order Magnus expan.)

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Analysis of Dynamical Decoupling

We'll need some formal background...

Decoherence: Isolated vs Open System Evolution

Isolated system: $H = H_S$

$$|\psi(t)\rangle = U_S(t)|\psi(0)\rangle \quad \dot{U}_S = -iH_S U_S \quad U_S(0) = I$$

equivalently: $|\psi(t)\rangle\langle\psi(t)| = U_S(t)|\psi(0)\rangle\langle\psi(0)|U_S^\dagger(t)$

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Open system: $H = H_S + H_B + H_{SB}$

$$\rho_{SB}(t) = U(t)\rho_{SB}(0)U^\dagger(t) \quad \dot{U} = -iHU \quad U(0) = I$$

$$\rho_S(t) = \text{Tr}_B \rho_{SB}(t)$$

\neq unitary transformation of $\rho_S(0)$
(except when there is a decoherence-free subspace)

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decoherence: $\|\rho_S(t) - |\psi(t)\rangle\langle\psi(t)|\| > 0$

which norm?

Kolmogorov Distance and Quantum Measurements (I)

Given two classical probability distributions $\{p_i^{(1)}\}$ and $\{p_i^{(2)}\}$, their deviation is measured by the Kolmogorov distance

$$D(p^{(1)}, p^{(2)}) \equiv \frac{1}{2} \sum_i |p_i^{(1)} - p_i^{(2)}|$$

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A quantum measurement can always be described in terms of a POVM (positive operator valued measure), i.e., a set of **positive** operators E_i satisfying $\sum_i E_i = I$, where i enumerates the possible measurement outcomes.

For a system initially in the state ρ , outcome i occurs with probability

$$p_i = \Pr(i|\rho) = \text{Tr}(\rho E_i)$$

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Thus quantum measurements produce classical probability distributions.

Consider two quantum states $\rho^{(1)} [= |\psi(t)\rangle\langle\psi(t)|]$ and $\rho^{(2)} [= \rho_S(t)]$

Kolmogorov Distance and Quantum Measurements (II)

Compare measurement outcomes of same POVM on $\rho^{(1)} [= |\psi(t)\rangle\langle\psi(t)|]$ and $\rho^{(2)} [= \rho_S(t)]$:

$$\text{Lemma: } \delta \equiv D(p_{\rho^{(1)}}, p_{\rho^{(2)}}) \leq \|\rho^{(1)} - \rho^{(2)}\|_{\text{Tr}}$$

$$\|A\|_{\text{Tr}} \equiv \text{Tr}\sqrt{A^\dagger A}$$

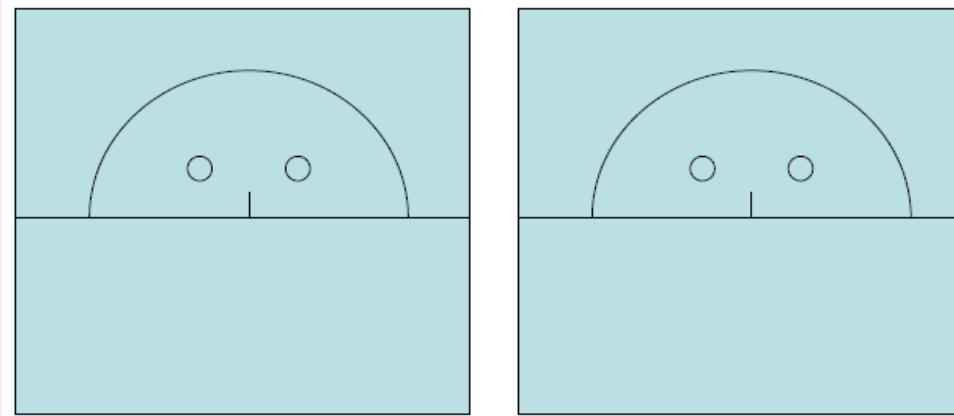
The bound is tight in the sense that it is saturated for the optimal measurement designed to distinguish the two states.

Partial trace decreases trace distance

Lemma: $\|\rho_S^{(1)} - \rho_S^{(2)}\|_{\text{Tr}} \leq \|\rho_{SB}^{(1)} - \rho_{SB}^{(2)}\|_{\text{Tr}}$

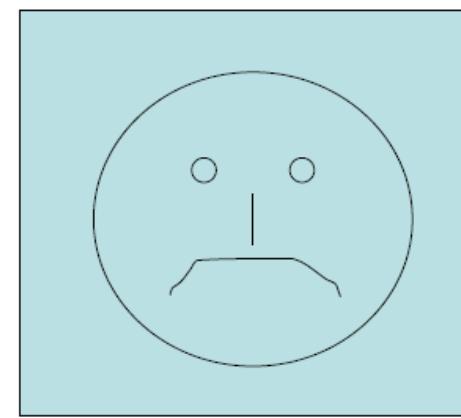
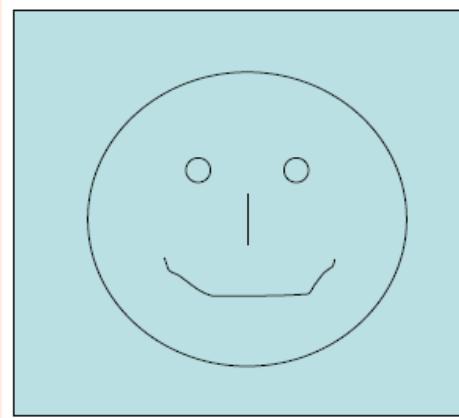
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Conclusion: we can compare dynamics of ideal and actual systems over the joint system-bath space.

Ideal vs Actual System Evolution

Ideal system: $H = H_S + H_B$

$$\rho_{SB}^{\text{ideal}}(t) = [U_S(t) \otimes U_B(t)]\rho_{SB}(0)[U_S^\dagger(t) \otimes U_B^\dagger(t)]$$

$$\dot{U}_{S/B} = -iH_{S/B}U_{S/B} \quad U_{S/B}(0) = I$$

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Distance:

$$\|\rho_{SB}(t) - \rho_{SB}^{\text{ideal}}(t)\|_{\text{Tr}} = \|V(t)\rho_{SB}(0)V^\dagger(t) - \rho_{SB}(0)\|_{\text{Tr}}$$

$$V(t) \equiv U_S^\dagger(t) \otimes U_B^\dagger(t)U(t) \equiv \exp[-itH_{\text{eff}}(t)]$$

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Lemma: $\|\rho_{SB}(t) - \rho_{SB}^{\text{ideal}}(t)\|_{\text{Tr}} \leq t\|H_{\text{eff}}(t)\|_\infty$

(follows from $\|e^{iA} - e^{iB}\|_\infty \leq \|A - B\|_\infty$)

Kolmogorov Distance Bound from Effective Hamiltonian

Lemma: $\delta \equiv D(p_{\rho^{(1)}}, p_{\rho^{(2)}}) \leq \|\rho^{(1)} - \rho^{(2)}\|_{\text{Tr}}$

(trace distance bounds
Kolmogorov distance)

Lemma: $\|\rho_S^{(1)} - \rho_S^{(2)}\|_{\text{Tr}} \leq \|\rho_{SB}^{(1)} - \rho_{SB}^{(2)}\|_{\text{Tr}}$

(partial trace decreases
distinguishability)

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Goal: reduce effective Hamiltonian.

Method: dynamical decoupling.

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How do we compute $H_{\text{eff}}(t)$?

The Magnus expansion

$$\dot{U} = -iH(t)U$$

$$U(t) = e^{-itH_{\text{eff}}(t)}$$

$$H_{\text{eff}}(t) = \frac{1}{t} \sum_{j=1}^{\infty} \Omega_j(t)$$

$$\Omega_1(t) = \int_0^t dt_1 H(t_1) \quad \Omega_2(t) = -\frac{i}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)]$$

Dynamical Decoupling against Arbitrary Interactions

“Symmetrizing group” of pulses $\{g_i\}$ and their inverses are applied in series:

$$(g_N^\dagger \mathbf{f} g_N) \cdots (g_2^\dagger \mathbf{f} g_2) (g_1^\dagger \mathbf{f} g_1) \approx \exp(-i\tau \sum_i g_i^\dagger H_{SB} g_i)$$

$\mathbf{f} \equiv \exp(-iH_{SB}\tau)$

first order Magnus expansion

Periodic DD: periodic repetition of the universal DD pulse sequence

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Choose the pulses so that:

$$H_{SB} \mapsto H_{\text{eff}}^{(1)} \equiv \sum_i g_i^\dagger H_{SB} g_i = 0 \quad \text{Dynamical Decoupling Condition}$$

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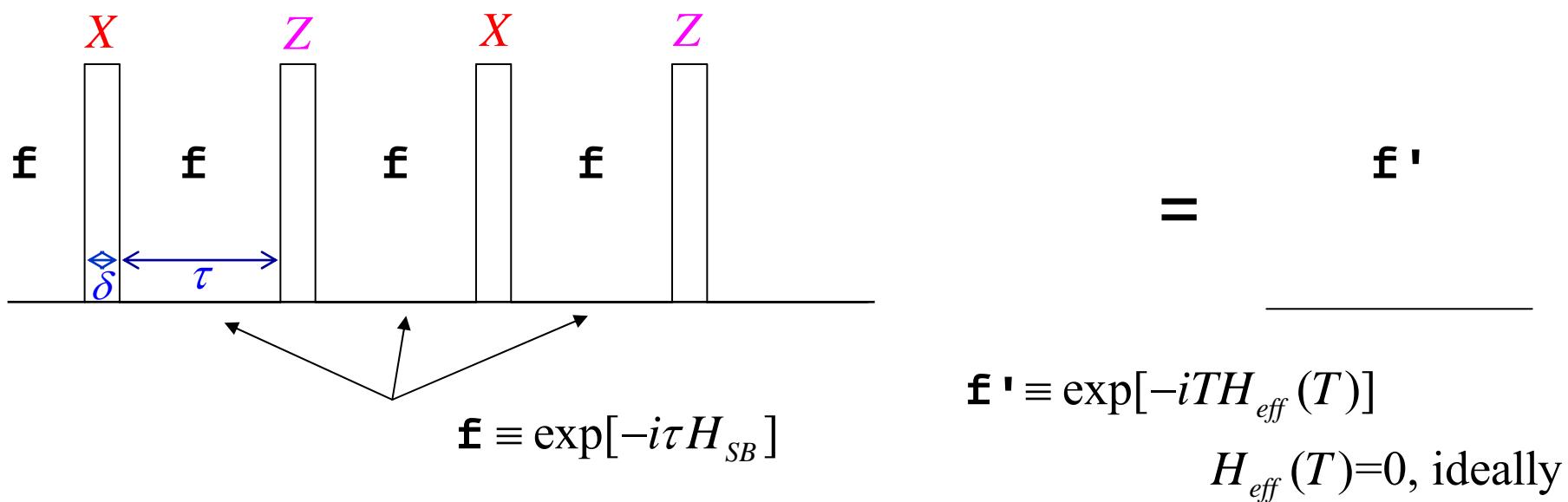
For a qubit the Pauli group $G=\{X, Y, Z, I\}$ (π pulses around all three axes) removes an arbitrary H_{SB} :

$$(X \mathbf{f} X) (Y \mathbf{f} Y) (Z \mathbf{f} Z) (I \mathbf{f} I) = \underline{\mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f}}$$

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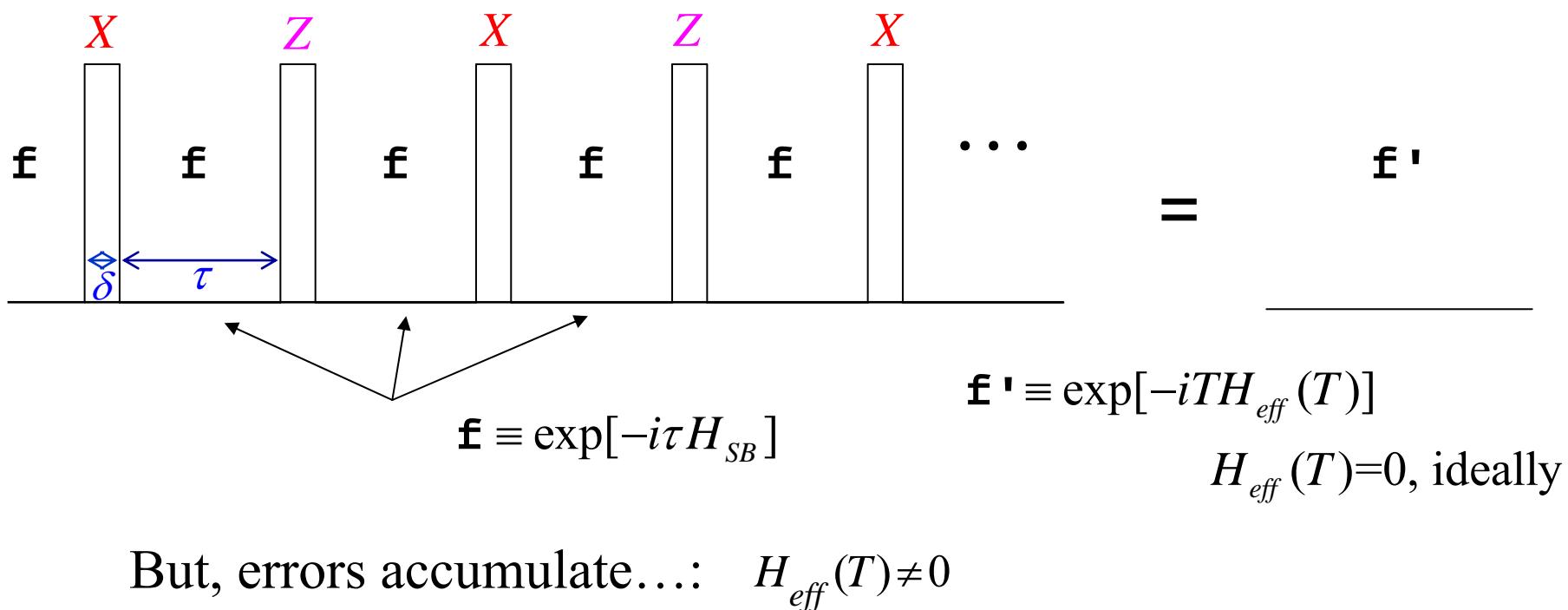
The Effective Hamiltonian

Another view of the universal decoupling sequence:



The Effective Hamiltonian

Another view of the universal decoupling sequence:



But, errors accumulate...: $H_{eff}(T) \neq 0$

DD as a Rescaling Transformation

$$J = \|H_{SB}\|_\infty$$

$$\beta = \|H_B\|_\infty$$

- Interaction terms are **rescaled** after the DD cycle

$$J = J^{(0)} \mapsto J^{(1)} \propto \max[\tau(J^{(0)})^2, \tau\beta J^{(0)}]$$

$$\beta \mapsto \beta + O((J^{(0)})^3 \tau^2)$$

- We need a mechanism to continue this

Concatenated Universal Dynamical Decoupling

Nest the universal DD pulse sequence into its own free evolution periods \mathbf{f} :

$\mathbf{p}(1) = \mathbf{x} \ f \quad \mathbf{z} \ f \quad \mathbf{x} \ f \quad \mathbf{z} \ f$

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$$p(n+1) = \mathbf{x} \ p(n) \mathbf{z} \ p(n) \mathbf{x} \ p(n) \mathbf{z} \ p(n)$$

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Level	Concatenated DD Series after multiplying Pauli matrices
1	$\mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f}$
2	$\mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{Y} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{f} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{Y} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f}$
3	$\mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{Y} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{f} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{Y} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{Y} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f}$ $\mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{Y} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{f} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{Y} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{Y} \mathbf{f} \mathbf{Z} \mathbf{f} \mathbf{X} \mathbf{f} \mathbf{Z} \mathbf{f}$

Length grows exponentially; how about error reduction?

Performance of Concatenated Sequences

The pulse interval is τ .

Apply a total of R^n pulses, where n is the concatenation level.

Recall:

$$\delta_{\text{actual,ideal}} \leq t \|H_{\text{eff}}(t)\|_\infty$$

For zero-width (ideal) pulses:

$$\delta_{\text{actual,ideal}} \leq c^n R^{n(n+3)/2} (J\tau)((J + \beta)\tau)(\beta\tau)^{n-1}$$

competition

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Thus there is an optimal concatenation level, at which:

$$\delta_{\text{actual,ideal}} \leq c(J\tau)((J + \beta)\tau)R^{-\frac{1}{2}(\xi^2 - \xi) + 2}$$

$$\text{where } \xi = -\log_R c(R(R - 1)\beta\tau)$$

The measurement outcome distribution of an open system of qubits can be made arbitrarily close to that of the corresponding closed system, using CDD.

- fitting the pieces together

Hybrid DD-DFS Quantum Computing

Computation

Problem: DD pulses interfere with computational pulses – they cancel everything!

How can they be reconciled?

- Need a **commuting** structure of pulses and computation.
- Solution:

Use **encoded qubits** from a **DFS**.

Logical gates over DFS generated by Heisenberg.

DD pulses acting as global X and Z are still a universal decoupling group, and commute with Heisenberg

Heisenberg Computation over DFS is Universal

- Heisenberg exchange interaction:

$$H_{\text{Heis}} = \sum_{i,j} J_{ij} (X_i X_j + Y_i Y_j + Z_i Z_j) \equiv \sum_{i,j} J_{ij} E_{ij}$$

- Universal over collective-decoherence DFS

[J. Kempe, D. Bacon, D.A.L., B. Whaley, Phys. Rev. A **63**, 042307 (2001)]

- Over 4-qubit DFS: $|0_L\rangle = \frac{1}{2}(|01\rangle - |10\rangle)(|01\rangle - |10\rangle)$

$$|1_L\rangle = \frac{1}{2\sqrt{3}}(2|0011\rangle + 2|1100\rangle - (|0110\rangle + |1001\rangle + |1010\rangle + |0101\rangle))$$

$$\bar{X} = -\frac{2}{\sqrt{3}}(E_{13} + \frac{1}{2}E_{12}) \quad \bar{Z} = -E_{12}$$

$e^{i\theta\bar{X}}$ and $e^{i\theta\bar{Z}}$ generate arbitrary single encoded qubit gates

CNOT involves 14 elementary steps (D. Bacon, Ph.D. thesis)

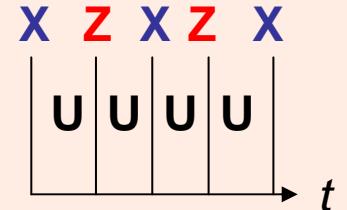
Universal Decoupling Group Commutes with Heisenberg

- n levels of concatenation, $N=4^n$ pulses
- Universal decoupling group on M (even) system-spins:

$$p(1) = \begin{matrix} X & U & Z & U & X & U & Z & U \end{matrix} \rightarrow e^{-i(\theta/N)H_{\text{gate}}}$$

$\downarrow \qquad \qquad \qquad \downarrow$

$$X_1 \cdots X_M \qquad \qquad \qquad Z_1 \cdots Z_M$$



$$p(2) = \begin{matrix} X & p(1)Z & p(1)X & p(1)Z & p(1) \end{matrix} \dots$$

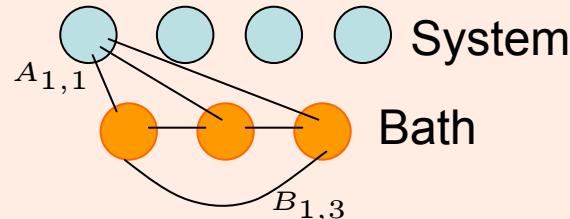
$$[H_{\text{Heis}}, X \text{ or } Z] = 0$$

Error Model: Electron qubits in GaAs, nuclear spin bath

- “Error Hamiltonian” (everything excluding DD):

$$H_e = \omega_S \sum_n \sigma_Z^{(n)} + \omega_B \sum_m \sigma_Z^{(m)} + \sum_{n < m} A_{n,m} \vec{\sigma}^{(n)} \cdot \vec{\sigma}^{(m)}$$

Heisenberg System-Bath (hyperfine coupling)

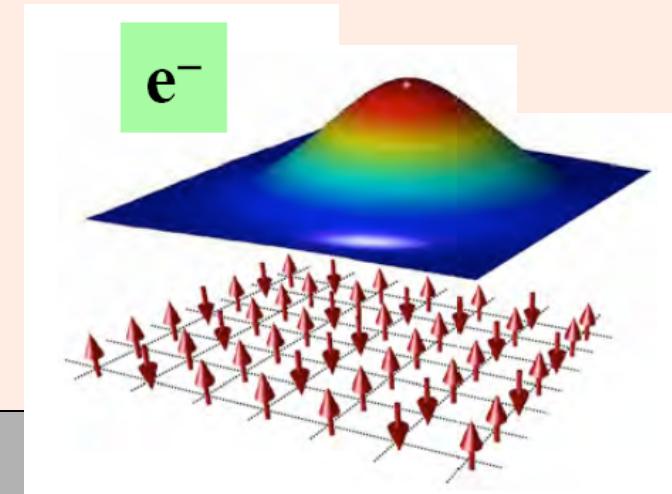


$$+ \sum_{n < m} B_{n,m} (\vec{\sigma}^{(n)} \cdot \vec{\sigma}^{(m)} - 3\sigma_X^{(n)}\sigma_X^{(m)})$$

Dipolar Bath-Bath

$$A_{n,m} = C/2^{d_{n,m}}, \text{ and } B_{n,m} = D/d_{n,m}^3$$

$$\beta = \sum_{n < m} B_{n,m} = 10 \text{ kHz}$$



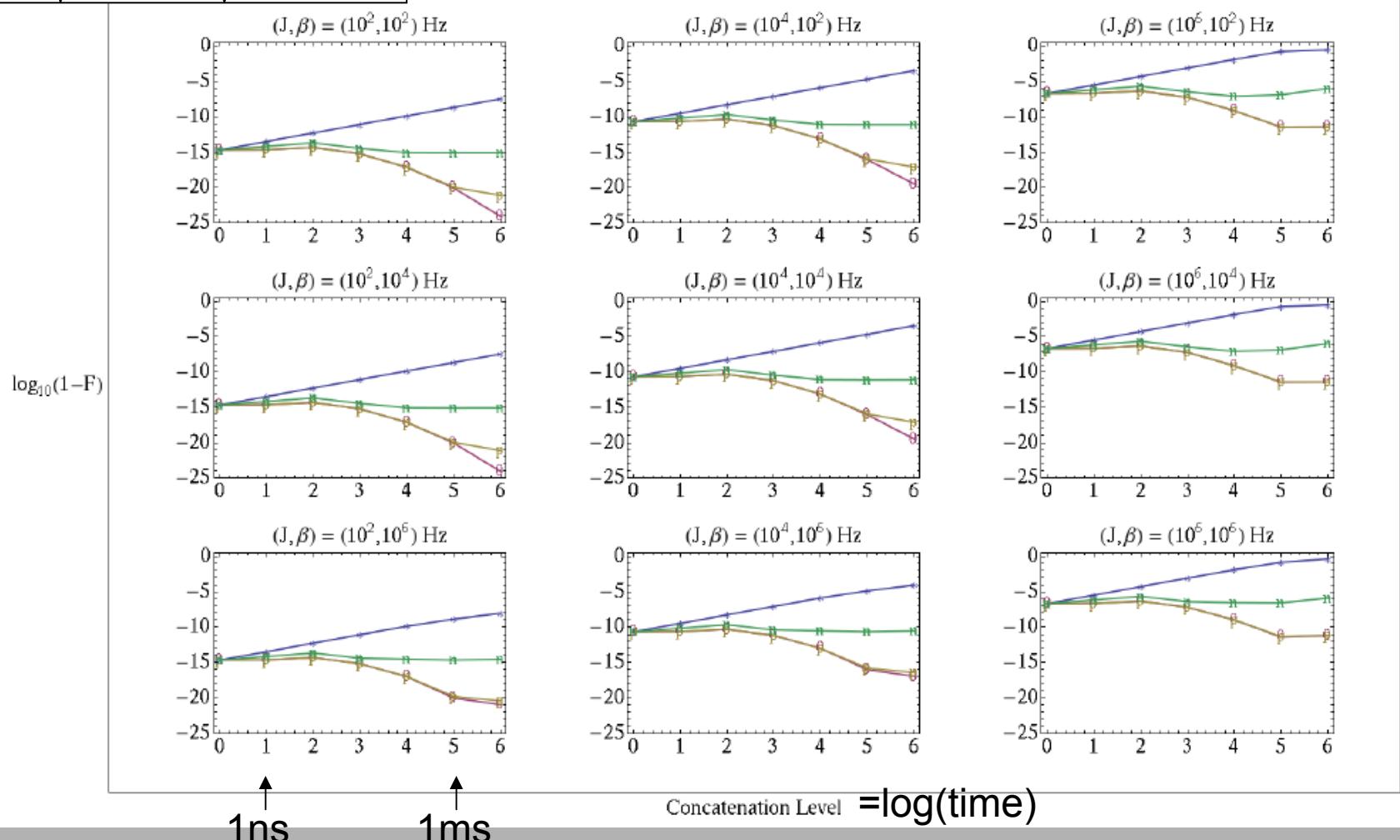
Model Assumptions for Exact Numerical Simulations

- Simulated discrete universal set: $\{\pi/8, \text{Hadamard}, \text{CPhase}\}$
- Single-encoded qubit gates: 4 system qubits, 6 bath qubits
- Encoded CPhase: 8 system qubits, 2 bath qubits
- Bath initialized in thermal equilibrium at zero K (equal superposition over all basis states)
- System initialized as $\frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle)$

Simulations for a protected logic gate: electron spin in a GaAs quantum dot

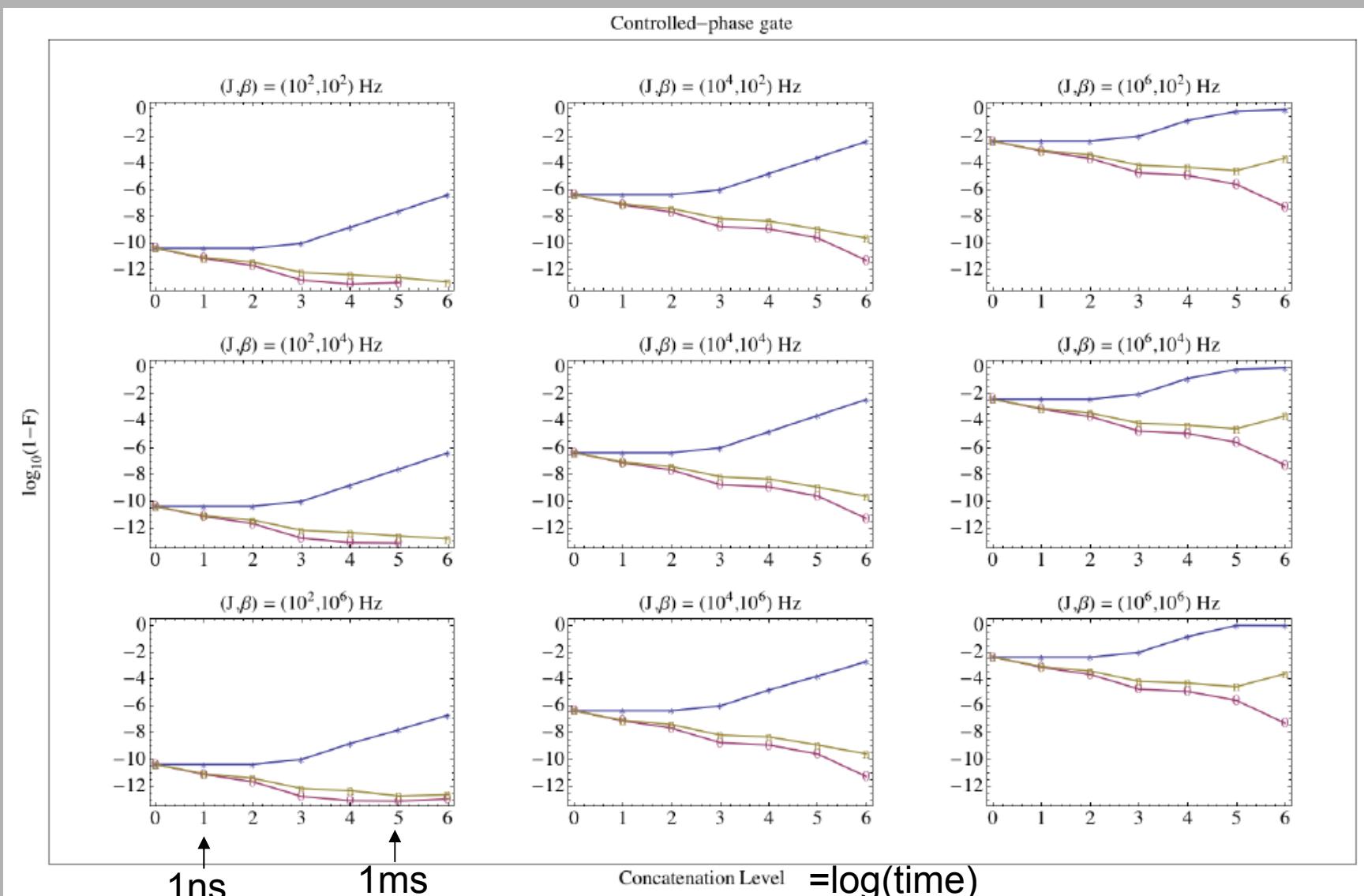
$$F = \left\| \sqrt{\rho_S^{\text{actual}}} \sqrt{\rho_S^{\text{ideal}}} \right\|_{\text{Tr}}$$

$\pi/8$ -gate



Gate implemented using Heisenberg interactions over 4 qubit DFS code.
 Pulse widths: 0, 1ps, 1ns

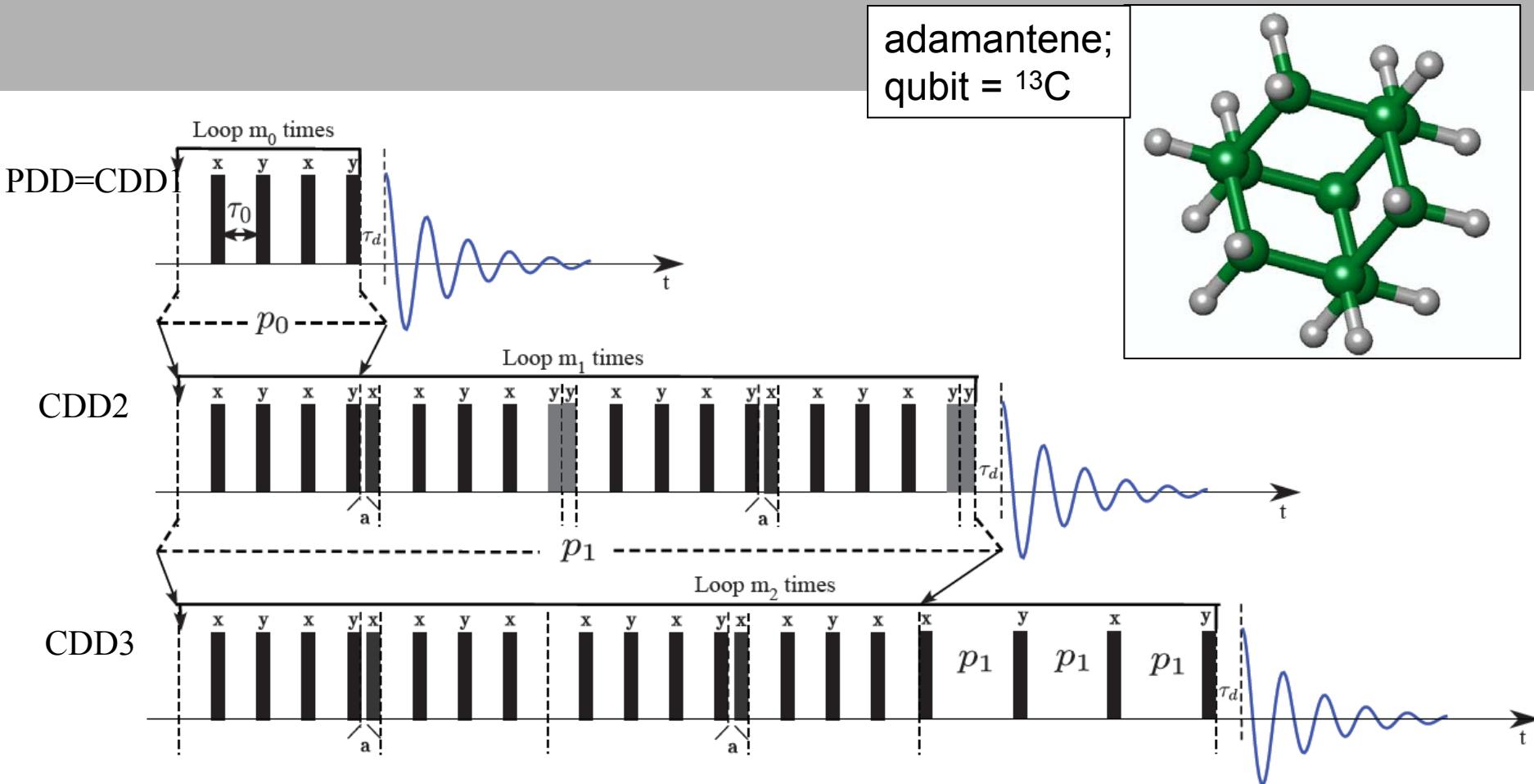
Simulations for a protected CPHASE gate: electron spin in GaAs quantum dot



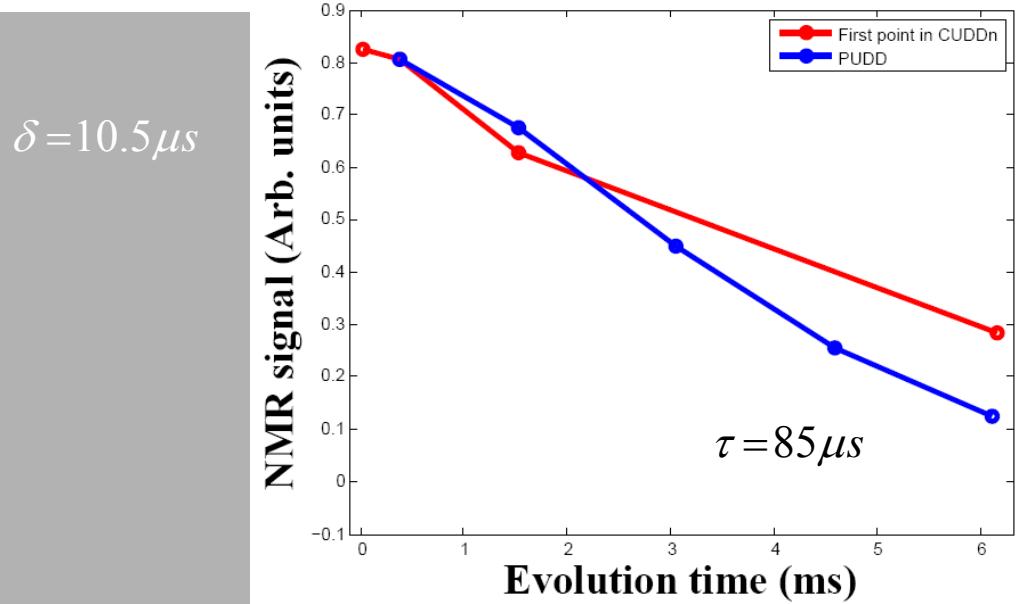
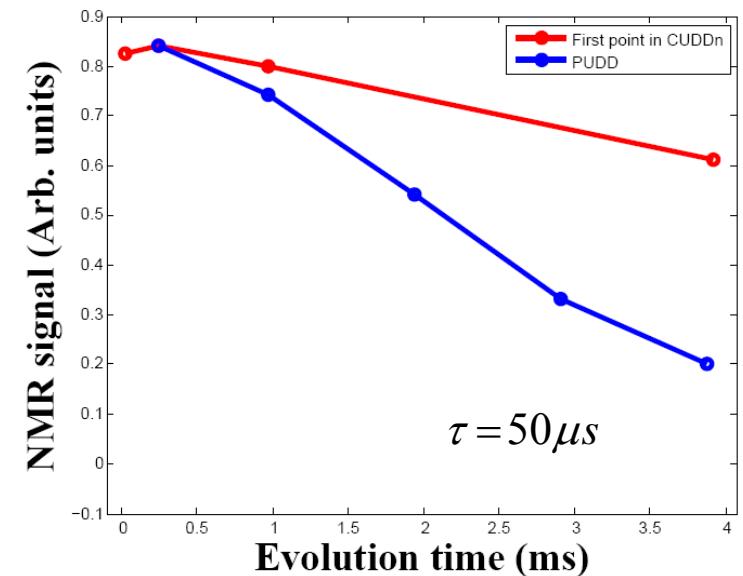
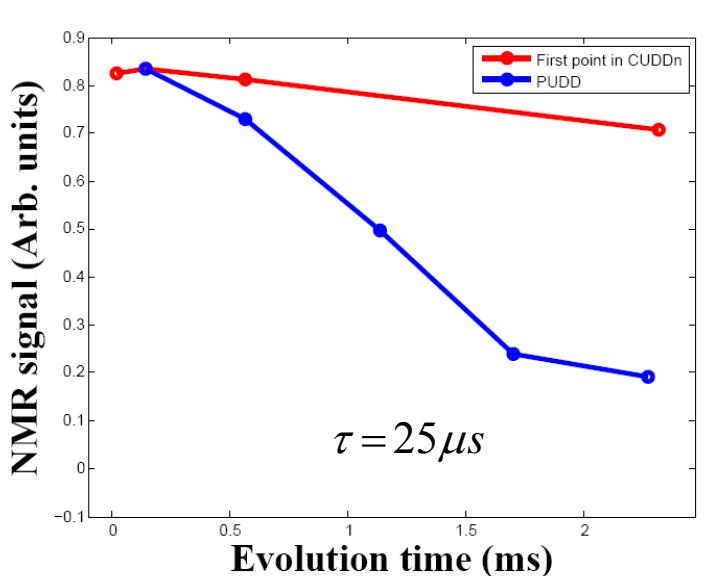
Gate implemented using Heisenberg interactions over two 4 qubit DFS codes. Pulse widths: 0, 1ps

Experiments

Experimental CDD on Adamantene (Dieter Suter)



CDD Results

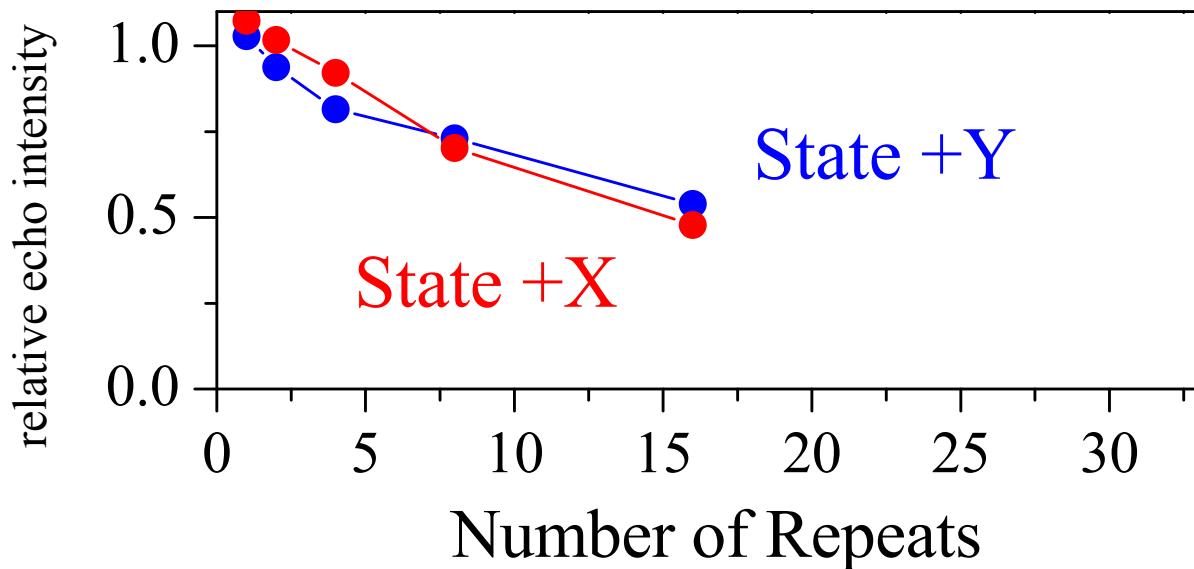


CDD for electron spin of ^{31}P donors in Si

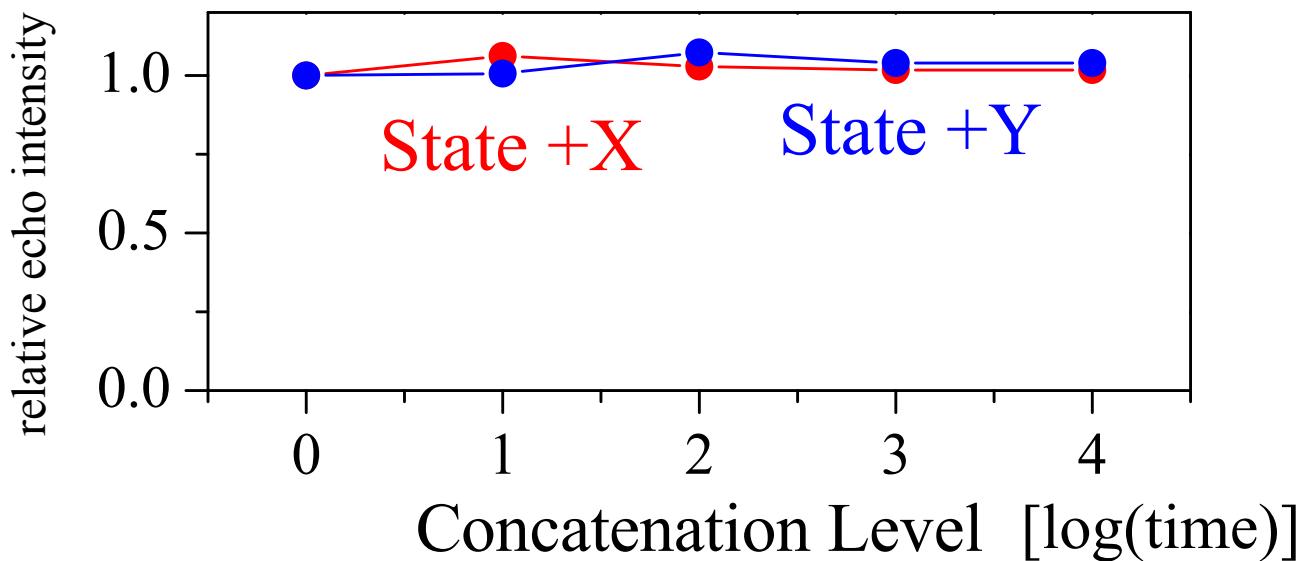
(Steve Lyon, Princeton)

1. Periodic
(XfZfXfZf)

$\delta=160\text{ns}$



2. Concatenated



Hybrid Q. Error Correction: The Big Picture

