Geometry of Schur maps and their application to random unitary channels

C. O'Meara

Department of Mathematics & Statistics
University of Guelph, Ontario, Canada

CQISC Toronto, 2009

Outline

Errors in QIT

Errors occur in the realization of gates in quantum information theory. There are two kinds of error mechanisms

1. Classical: phase shift
2. Quantum: spontaneous decay

If the error occurring is classical, the mathematical formulation is that it can be described by a mixture of unitary maps.

Errors in QIT

Errors occur in the realization of gates in quantum information theory. There are two kinds of error mechanisms

1. Classical: phase shift
2. Quantum: spontaneous decay

If the error occurring is classical, the mathematical formulation is that it can be described by a mixture of unitary maps.

Errors in QIT

Errors occur in the realization of gates in quantum information theory. There are two kinds of error mechanisms

1. Classical: phase shift
2. Quantum: spontaneous decay

If the error occurring is classical, the mathematical formulation is that it can be described by a mixture of unitary maps.

Let the linear map $\Phi : \mathbb{M}_n \longmapsto \mathbb{M}_n$, then

- Φ is Hermitian if it maps Hermitian matrices to Hermitian matrices
- Φ is completely positive (CP) if $\Phi \otimes \mathbb{I}_n$ is positive $\forall n \in \mathbb{N}$
- Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$ is positive

A linear map Φ that is positive (or respectively CP);

- Φ is stochastic if it is trace preserving (TP); Tr[$\Phi(A)$]=Tr[A], $\forall A \in \mathbb{M}_n$
- Φ is doubly stochastic if $\Phi(\mathbb{I}) = \mathbb{I}$ and is TP

A linear map Φ that is CPT is called a Quantum Channel.

Let the linear map $\Phi : \mathbb{M}_n \longmapsto \mathbb{M}_n$, then

- Φ is Hermitian if it maps Hermitian matrices to Hermitian matrices
- Φ is completely positive (CP) if $\Phi \otimes \mathbb{I}_n$ is positive $\forall n \in \mathbb{N}$
- Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$ is positive

A linear map Φ that is positive (or respectively CP);

- Φ is stochastic if it is trace preserving (TP); Tr[$\Phi(A)$]=Tr[A], $\forall A \in \mathbb{M}_n$
- Φ is doubly stochastic if $\Phi(\mathbb{I}) = \mathbb{I}$ and is TP

A linear map Φ that is CPT is called a <u>Quantum Channel</u>.

Choi-Kraus Representation

A linear map $\Phi : \mathbb{M}_n \longmapsto \mathbb{M}_n$ is CP iff it admits the following form

$$\Phi(A) = \sum_{i}^{R} V_i A V_i^* \tag{1}$$

For completeness; $\sum_{i=1}^{R} V_i^* V_i = \mathbb{I}$. The map Φ is unital and hence UCPT if

$$\Phi(\mathbb{I}) = \sum_{i}^{R} V_i V_i^* = \mathbb{I}$$

The set of UPCT maps CP_I is a compact convex set and by Krein-Milman is therefore represented as the convex hull of its extreme points.

Convexity

Convex Set

A set C$\in \mathbb{C}$ (or $\mathbb{R}$) is convex if $\forall\ x_1, x_2 \in C$,

$$\lambda x_1 + (1-\lambda) x_2 \in C$$

Face

F$\subseteq$ C is a face if

$$F = C \cap H$$

where H is the separating hyperplane.

Extreme point

A face F of dimension 0 is an extreme point of C.

Convexity

Convex Set

A set C$\in \mathbb{C}$ (or $\mathbb{R}$) is convex if $\forall\ x_1, x_2 \in C$,

$$\lambda x_1 + (1-\lambda)x_2 \in C$$

Face

F$\subseteq C$ is a face if

$$F = C \cap H$$

where H is the separating hyperplane.

Extreme point

A face F of dimension 0 is an extreme point of C.

Convexity

Convex Set

A set C$\in \mathbb{C}$ (or $\mathbb{R}$) is convex if $\forall\ x_1, x_2 \in C$,

$$\lambda x_1 + (1-\lambda)x_2 \in C$$

Face

F$\subseteq$ C is a face if

$$F = C \cap H$$

where H is the separating hyperplane.

Extreme point

A face F of dimension 0 is an extreme point of C.

Examples

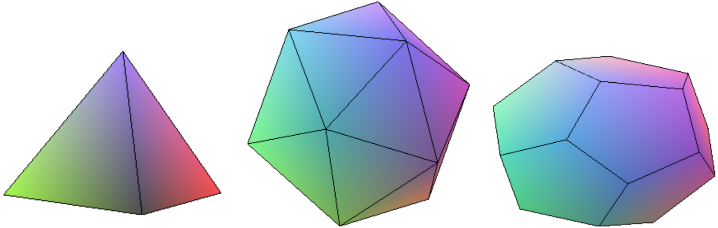

Fig.1. Platonic solids

Now we can discuss an important concept...

Generalization of Choi's Theorem

Choi's Theorem

The map $\Phi(A) = \sum_i^R V_i A V_i^*$ is extremal in CP_K iff the following hold

- $\sum_i^R V_i V_i^* = K$
- $\{V_i V_j^*\}$ $1 \leq i, j \leq R$ is a L.I set

Corollary to L.S. Theorem

The map $\Phi(A) = \sum_i^R V_i A V_i^*$ is extremal in CP_I iff the following hold

- $\sum_i^R V_i V_i^* = \mathbb{I}$
- $\sum_i^R V_i^* V_i = \mathbb{I}$
- $\{V_i V_j^*\}$ and $\{V_i^* V_j\}$ $1 \leq i, j \leq R$ are L.I sets

Generalization of Choi's Theorem

Choi's Theorem

The map $\Phi(A) = \sum_i^R V_i A V_i^*$ is extremal in CP_K iff the following hold

- $\sum_i^R V_i V_i^* = K$
- $\{V_i V_j^*\}$ $1 \leq i,j \leq R$ is a L.I set

Corollary to L.S. Theorem

The map $\Phi(A) = \sum_i^R V_i A V_i^*$ is extremal in CP_I iff the following hold

- $\sum_i^R V_i V_i^* = \mathbb{I}$
- $\sum_i^R V_i^* V_i = \mathbb{I}$
- $\{V_i V_j^*\}$ and $\{V_i^* V_j\}$ $1 \leq i,j \leq R$ are L.I sets

Question:

Given a UCPT map, how do we determine whether the map can be decomposed as a convex combination of unitary maps?

i.e

$$A \longrightarrow \Phi(A) = \sum_{i}^{R} p_i U_i A U_i^*$$

UCPT maps that can be decomposed in this way are called random unitary maps.

Any 2 x 2 Hermitian matrix $H \in \mathbb{C}^2$can be represented as a real linear combination of the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ and the identity;

$$\Rightarrow H = span\{I, \sigma_x, \sigma_y, \sigma_z\}$$

The requirement that for a density matrix, $Tr(\rho) = 1$, forces the coefficient of the first basis vector, the identity I, be $\frac{1}{2}$. This in effect means we can expresses the 4 point basis with a 3-dimensional real vector subspace.

An arbitrary density matrix ρ for a qubit can be represented by

$$\rho = \frac{\mathbb{I} + \mathbf{r} \cdot \sigma}{2}$$

$$\rho = \frac{1}{2} \begin{bmatrix} 1 + r_z & r_x - i r_y \\ r_x + i r_y & 1 - r_z \end{bmatrix} \tag{2}$$

where $\mathbf{r}$ is known as the *Bloch vector*. The set of permissible density matrices ρ are represented by the ball $\|\mathbf{r}\| \leq 1$, the *Bloch Sphere*.

Example

Play

Fig.2. The Bloch Sphere representation of a single qubit state before and after the bit flip error

It turns out random unitary channels have a nice geometric interpretation in terms of the bloch sphere...

What is the bloch vector of a density matrix representing a R.U map?

It is a well known result that using the maximally entangled state vector

$$|I\rangle := \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle$$

we obtain the Choi matrix via the Jamiolkowski isomorphism;

$$J(\Phi) = \rho_\Phi = (\Phi \otimes \mathbb{I})(|I\rangle\langle I|) \tag{3}$$

where $\mathbb{I}$ is the identity map.

So using $\Phi(A) = \sum_i^R p_i U_i A U_i^*$, we obtain that the map Φ is a random unitary map if its Choi matrix has the form;

$$\rho_\Phi = \sum_i^R p_i (U_i \otimes \mathbb{I})(|I\rangle\langle I|)(U_i^\dagger \otimes \mathbb{I}) = \sum_i^R p_i |\psi\rangle\langle\psi|$$

i.e. A convex combination of maximally entangled pure states.

$\Rightarrow$ random unitary maps are in fact a convex set with rank 1 UCPT maps as the extreme points.

$\Rightarrow$ Denote this set R_U.

So using $\Phi(A) = \sum_i^R p_i U_i A U_i^*$, we obtain that the map Φ is a random unitary map if its Choi matrix has the form;

$$\rho_\Phi = \sum_i^R p_i (U_i \otimes \mathbb{I})(|I\rangle\langle I|)(U_i^\dagger \otimes \mathbb{I}) = \sum_i^R p_i |\psi\rangle\langle\psi|$$

i.e. A convex combination of maximally entangled pure states.

$\Rightarrow$ random unitary maps are in fact a convex set with rank 1 UCPT maps as the extreme points.

$\Rightarrow$ Denote this set R_U.

So using $\Phi(A) = \sum_i^R p_i U_i A U_i^*$, we obtain that the map Φ is a random unitary map if its Choi matrix has the form;

$$\rho_\Phi = \sum_i^R p_i (U_i \otimes \mathbb{I})(|I\rangle\langle I|)(U_i^\dagger \otimes \mathbb{I}) = \sum_i^R p_i |\psi\rangle\langle\psi|$$

i.e. A convex combination of maximally entangled pure states.

$\Rightarrow$ random unitary maps are in fact a convex set with rank 1 UCPT maps as the extreme points.

$\Rightarrow$ Denote this set R_U.

Furthermore, Audenaert and Scheel proved that a pure state is maximally entangled and hence an extreme point of R_U iff

$$\begin{aligned} |\psi\rangle\langle\psi| &= (\mathbb{I}/\sqrt{d} \otimes \mathbb{I})|I\rangle\langle I|(\mathbb{I} \otimes \mathbb{I}/\sqrt{d}) \\ \rightarrow Tr_B|\psi\rangle\langle\psi| &= \frac{\mathbb{I}}{d} \end{aligned}$$

That is, it is equivalent to saying that the reduced bloch vector is the zero vector;

$$\begin{aligned} Tr[\tau_i Tr_B|\psi\rangle\langle\psi|] &= Tr[\tau_i Tr\rho] = 0, \qquad 1 \leq i \leq d^2 - 1 \\ \text{or} &= \langle\psi|\tau_i \otimes \mathbb{I}|\psi\rangle = 0, \qquad 1 \leq i \leq d^2 - 1 \end{aligned}$$

Furthermore, Audenaert and Scheel proved that a pure state is maximally entangled and hence an extreme point of R_U iff

$$\begin{aligned} |\psi\rangle\langle\psi| &= (\mathbb{I}/\sqrt{d}\otimes\mathbb{I})|I\rangle\langle I|(\mathbb{I}\otimes\mathbb{I}/\sqrt{d}) \\ \rightarrow Tr_B|\psi\rangle\langle\psi| &= \frac{\mathbb{I}}{d} \end{aligned}$$

That is, it is equivalent to saying that the reduced bloch vector is the zero vector;

$$\begin{aligned} Tr[\tau_i Tr_B|\psi\rangle\langle\psi|] &= Tr[\tau_i Tr\rho] = 0, \qquad 1 \leq i \leq d^2 - 1 \\ \text{or} &= \langle\psi|\tau_i\otimes\mathbb{I}|\psi\rangle = 0, \qquad 1 \leq i \leq d^2 - 1 \end{aligned}$$

Schur map

A UCPT map Φ is a Schur map iff it has the form

$$\Phi_C : A \longrightarrow A \circ C$$

which operates as;

$$[a_{ij}] \longrightarrow [a_{ij} c_{ij}]$$

Φ_C is CP iff C is PSD. Φ_C is unital iff Φ_C is TP iff $C_{ii} = 1$

A map $\Phi_C(A)$ that is CP and unital corresponds to C being a correlation matrix.

Nomenclature

The set of Schur maps and correlation matrices both form compact convex sets we denote D_Φ and ε_{nxn}, respectively.

Schur map

A UCPT map Φ is a Schur map iff it has the form

$$\Phi_C : A \longrightarrow A \circ C$$

which operates as;

$$[a_{ij}] \longrightarrow [a_{ij}c_{ij}]$$

Φ_C is CP iff C is PSD. Φ_C is unital iff Φ_C is TP iff $C_{ii} = 1$
A map $\Phi_C(A)$ that is CP and unital corresponds to C being a correlation matrix.

Nomenclature

The set of Schur maps and correlation matrices both form compact convex sets we denote D_Φ and ε_{nxn}, respectively.

Schur map

A UCPT map Φ is a Schur map iff it has the form

$$\Phi_C : A \longrightarrow A \circ C$$

which operates as;

$$[a_{ij}] \longrightarrow [a_{ij}c_{ij}]$$

Φ_C is CP iff C is PSD. Φ_C is unital iff Φ_C is TP iff $C_{ii} = 1$
A map $\Phi_C(A)$ that is CP and unital corresponds to C being a correlation matrix.

Nomenclature

The set of Schur maps and correlation matrices both form compact convex sets we denote D_Φ and ε_{nxn}, respectively.

Correlation Matrices

Describe correlation between two spin random variables.

	X	Y
X	1	
Y		1

Fig.3. 2x2 correlation describing the random variables {X,Y}

Convex structure of ε_{3x3}

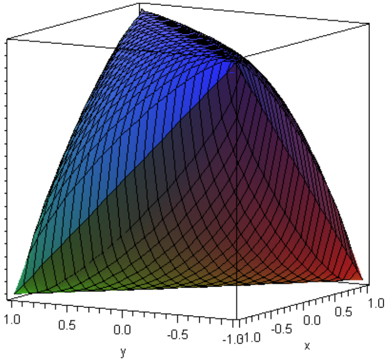

Fig.4. Convex structure of the set of 3x3 real correlation matrices we call ε_{3x3}

It is useful to consider the result proved by Landau and Streater which states that;

Landau-Streater

a) If Φ is a 3x3 Schur map, then it is a convex combination of rank 1 Schur complex maps.
b) Any rank 2 UCPT map is unitarily equivalent to a Schur map and hence is a convex combo of rank 1 complex Schur maps.

Which brings us to the first result;

Theorem 1

The UCPT Schur map Φ_C is extremal in the set of UCPT maps CP_I iff the correlation matrix C is extremal in ε_{nxn}.

It is useful to consider the result proved by Landau and Streater which states that;

Landau-Streater

a) If Φ is a 3x3 Schur map, then it is a convex combination of rank 1 Schur complex maps.
b) Any rank 2 UCPT map is unitarily equivalent to a Schur map and hence is a convex combo of rank 1 complex Schur maps.

Which brings us to the first result;

Theorem 1

The UCPT Schur map Φ_C is extremal in the set of UCPT maps CP_I iff the correlation matrix C is extremal in ε_{nxn}.

It is useful to consider the result proved by Landau and Streater which states that;

Landau-Streater

a) If Φ is a 3x3 Schur map, then it is a convex combination of rank 1 Schur complex maps.
b) Any rank 2 UCPT map is unitarily equivalent to a Schur map and hence is a convex combo of rank 1 complex Schur maps.

Which brings us to the first result;

Theorem 1

The UCPT Schur map Φ_C is extremal in the set of UCPT maps CP_I iff the correlation matrix C is extremal in ε_{nxn}.

Corollary 1

There exists an isomorphism $f : C \longrightarrow \Phi_C \; \forall$ C$\in \varepsilon_{nxn}$.

This finally brings us to the connection to the set of random unitary channels, R_U.

Theorem (Bhat, Pati, Sunder)

Let $\Phi_C \in \mathbb{C}$ be a Schur map. Φ_C is a random unitary channel iff $C = \sum_i^r p_i W_i$
where each $W_i = v_i v_i^*$, a rank 1 correlation matrix.

Corollary 1

There exists an isomorphism $f : C \longrightarrow \Phi_C \; \forall$ C$\in \varepsilon_{nxn}$.

This finally brings us to the connection to the set of random unitary channels, R_U.

Theorem (Bhat, Pati, Sunder)

Let $\Phi_C \in \mathbb{C}$ be a Schur map. Φ_C is a random unitary channel iff
$C = \sum_i^r p_i W_i$
where each $W_i = v_i v_i^*$, a rank 1 correlation matrix.

Idea of Proof: Follows from the diagonal invariance of the Schur map that

$$
\begin{aligned}
\Phi_C(A)_{jk} = C_{jk} \circ A_{jk} &= \sum_{\alpha}^{r} V_{ij}^{\alpha} A_{im} V_{mk}^{\alpha *} \\
\Rightarrow C_{jk}\delta_{ij}\delta_{mk} &= \sum_{\alpha}^{r} V_{ij}^{\alpha} V_{mk}^{\alpha *} \\
\Rightarrow \sum_{\alpha}^{r} |V_{ij}^{\alpha}|^2 &= 0 \\
\Rightarrow C_{jk} &= \sum_{\alpha}^{r} \lambda_j^{\alpha} \bar{\lambda}_k^{\alpha}
\end{aligned}
$$

i.e. C is composed of the rank 1 matrices formed by the diagonal vector elements of the Kraus operators.

BTS Cont'd

Since a random unitary operator has the form

$$\Phi(A) = \sum_{i}^{R} p_i U_i A U_i^*$$

Using the fact that each Kraus operator is diagonal for a Schur map we find that;

$$C_{jk} = \sum_{\alpha}^{r} p_\alpha \lambda_j^\alpha \bar{\lambda}_k^\alpha \quad \square$$

Corollary 2

For d≤3; $U_{\varepsilon(n)} = F(R_U)$
For d≥4; $U_{\varepsilon(n)r1} = F(R_U)$

where $U_{\varepsilon(n)r1}$ is the convex hull of only the rank 1 Schur maps.

BTS Cont'd

Since a random unitary operator has the form

$$\Phi(A) = \sum_{i}^{R} p_i U_i A U_i^*$$

Using the fact that each Kraus operator is diagonal for a Schur map we find that;

$$C_{jk} = \sum_{\alpha}^{r} p_\alpha \lambda_j^\alpha \bar{\lambda}_k^\alpha \quad \square$$

Corollary 2

For d≤3; $U_{\varepsilon(n)} = F(R_U)$
For d≥4; $U_{\varepsilon(n)r1} = F(R_U)$

where $U_{\varepsilon(n)r1}$ is the convex hull of only the rank 1 Schur maps.

The Schur map Φ_C is CP Hermitian if and only if the correlation matrix $C \in \mathbb{R}$. What kind of generalizations can we make about Hermitian quantum maps?

Theorem 2

Every 2x2 Hermitian UCPT map Φ can be expressed as;

$$\Phi_H(A) = \sum_i^r p_i R_i(A) R_i$$

where each R_i is a Hermitian unitary matrix.

NOTE: This satisfies the result that all 2x2 qubit maps are precisely the set of random unitary maps as reflection matrices are both Hermitian and unitary.

The Schur map Φ_C is CP Hermitian if and only if the correlation matrix $C \in \mathbb{R}$. What kind of generalizations can we make about Hermitian quantum maps?

Theorem 2

Every 2x2 Hermitian UCPT map Φ can be expressed as;

$$\Phi_H(A) = \sum_i^r p_i R_i(A) R_i$$

where each R_i is a Hermitian unitary matrix.

NOTE: This satisfies the result that all 2x2 qubit maps are precisely the set of random unitary maps as reflection matrices are both Hermitian and unitary.

More Bloch sphere interpretation

This form of quantum map effects the Bloch sphere in an interesting way...
$\Rightarrow$ Only a compression with no rotation to the Bloch vector.

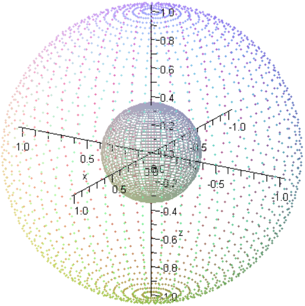

Fig.5. Convex combination of the Pauli Channels with $p_i = 1/3$

Summary

- Random unitary channels describe classical errors in information processing.
- Schur maps provide a useful geometric picture in analyzing mathematical properties of random unitary channels

- Outlook
  - Examine the more general notion of Hermitian maps for $d > 2$
  - Look at other applications of Schur maps

Special Thanks

- Prof. Rajesh Pereira
  Department of Mathematics and Statistics
  University of Guelph
- Prof. David Kribs
  Department of Mathematics and Statistics
  University of Guelph
- Fields Institute

Thank You!