The Higher Rank Numerical Range and its Implications in Quantum Data Error Correction

Canadian Quantum Information Student Conference
Fields Institute, Toronto, Canada

Nishan Mudalige

22nd August, 2009

Table of Contents I

Table of Contents II

History

History

History

- Historically, the classical numerical range has been studied extensively by mathematicians interested in the areas of functional analysis and matrix analysis [2].

- Originally *Toeplitz* and *Hausdorff* called it the *Wetvorrat* of a bilinear form, the Russian community has referred to it as the *Hausdorff domain*, *Murnaghan* called it many things, eventually settling on calling it the *field of values* and finally *Marshall Stone* in his influential book on operator theory cemented the name *numerical range* [2].

- Even now there is a fair degree of interest in the classical numerical range, in particular in the areas such as C^*-algebras, operator theory, dilation theory, matrix analysis etc. [2].

History Ctd...

- However, only recently has there been any interest given to the higher-rank numerical range.
- While researchers have bee studying quantum information theory, they have also been making significant contributions with ground breaking mathematics
- One of the most recent discoveries is the *higher-rank numerical range*. By **'recent'**, we mean **within the last 5 years!**
- The higher rank numerical range was first studied in two papers co-authored by M.D.Choi, D.W.Kribs and K.Życkowski. It was only during a conference in Calgary in 2006 that they announced this amazing discovery to the world. They are the first noted people to present the higher-rank numerical range.

History Ctd...

- This got the scientific community interested in expanding on the researchers' initial findings.
- Even more recently, the connection between the higher-rank numerical range and quantum data error correction has been established, and this is why it is of particular interest to us.
- This area is so recent that there has not even been a single textbook published on it. The resources available at present are about 15 - 20 papers (Most of which are written by authors that are present at this conference) and another handful of papers awaiting publishing.

History Ctd...

- So there really isn't a history of the higher-rank numerical range. This interesting discovery started with our professors and it continues with us!
- We are what will be the history of the higher-rank numerical range!

Introduction

Introduction

Notation

- $\mathcal{H}$ - A (finite dimensional) Hilbert space
- $\mathcal{L}(\mathcal{H})$ - The space of linear operators of Hilbert space $\mathcal{H}$
- $\mathbb{C}$ - The field of complex numbers
- P - A projection operator
- H - A Hermitian Operator
- U - A Unitary operator
- T - A normal operator
- A - An arbitrary operator
- $\Lambda_k(A)$ - The rank-k numerical range of operator A, $k \in \mathbb{Z}^+$

The Numerical Range

- The (classical) numerical range of an operator A is the set of complex numbers given by:

$$W(A) = \{< Ax, x >, x \in \mathcal{H}, ||x|| = 1\}$$

- Geometrically, it can be considered as the intersection of closed half-planes given by:

$\{\lambda \in \mathbb{C} | e^{i\theta}\lambda + e^{-i\theta}\bar{\lambda} \leq \lambda_1(e^{i\theta}A + e^{-i\theta}A^*)\}$ where $\theta \in [0, 2\pi)$ [4]

- This brings us to the following important theorem:

Theorem (Toeplitz – Hausdorff)

The numerical range of an operator is convex.

Preliminary Definitions

- Before we introduce the higher rank numerical range, recall the following definitions:

Definition (Dimension)

The **dimension** of a (finite-dimensional) complex Hilbert space $\mathcal{H}$ is the number of vectors that span any basis of $\mathcal{H}$.

Definition (Rank)

Let $M : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear transformation from Hilbert space $\mathcal{H}_1$ to Hilbert space $\mathcal{H}_2$. The *dimension of the range* of M is called the **rank of M**.

Preliminary Definitions Ctd...

Definition (Projection)

Let $P \in \mathcal{L}(\mathcal{H})$ for Hilbert Space $\mathcal{H}$. P is called a **projection** if:

$$P = P^2 = P^*$$

- A $rank - k$ projection P (more precisely a $rank - k$ *orthogonal projection operator* P) in a Hilbert space $\mathcal{H}$ of dimension N has the form:

$$P = \begin{bmatrix} 1_k & 0_{k\times(N-k)} \\ 0_{(N-k)\times k} & 0_{(N-k)\times(N-k)} \end{bmatrix}$$

The Higher Rank Numerical Range

Definition

Let k be a positive integer. Let $\mathcal{H}$ be a finite dimensional Hilbert space with dimension at least k; and let operator $A \in \mathcal{L}(\mathcal{H})$.

The *rank* $- k$ numerical range of A is defined as:

$$\Lambda_k(A) := \{\lambda \in \mathbb{C} | PAP = \lambda A\}$$

for some *rank* $- k$ orthogonal projection P.

- Notice the similarity to the familiar eigenvalue problem.
- Elements of $\Lambda_k(A)$ are called "compression values".
- When $k = 1$, we get $W(A)$, which is the *classical numerical range* of A.

Geometry of The Higher Rank Numerical Range

Geometry of The Higher Rank Numerical Range

Geometric Representation

- The compression values are all complex numbers, and take the form of $x + yi$ where $x, y \in \mathbb{R}$ and $i^2 = -1$.
- As such, we can represent the higher rank numerical range on the complex plane.
- The geometric representation of $\Lambda_k(A)$ will depend on the nature of operator the operator A.
- For this talk, we will discuss the geometry of 3 types of operators:
  1. Hermitian operators
  2. Unitary operators
  3. Normal operators

Geometric Properties

- The following properties were discovered by Choi, kribs and Życkowski:

1. $\Lambda_k(\alpha A + \beta I) = \alpha \Lambda_k(A) + \beta \quad \forall \alpha, \beta \in \mathbb{C}$
2. $\Lambda_k(A^*) = \overline{\Lambda_k(A)}$
3. $\Lambda(A) \subseteq \Lambda_k(\text{Re } A) + i\Lambda_k(\text{Im } A)$
4. $\Lambda_k(A \oplus B) \supseteq \Lambda_k(A) \cup \Lambda_k(B)$
5. $\Lambda_{k_1+k_2}(A \oplus B) \supseteq \Lambda_{k_1}(A) \cap \Lambda_{k_2}(B)$
6. $\Lambda_1(A) \supseteq \Lambda_2(A) \supseteq \Lambda_3(A) \supseteq \ldots \supseteq \Lambda_N(A)$

Geometric Properties Ctd...

- We will prove item 6 as it will become more evident as this talk progresses.
- To do this, we make use of the following property of projections:

Property

Let P_1 be a projection onto Hilbert space $\mathcal{H}_1$ and let P_2 be a projection onto Hilbert space $\mathcal{H}_2$, where $\mathcal{H}_2 \subseteq \mathcal{H}_1$. Then:

$$P_1 P_2 = P_2 P_1 = P_2$$

is a projection onto $\mathcal{H}_1 \cap \mathcal{H}_2$.

Proof of Property 6

- We wish to prove that for $A \in \mathcal{L}(\mathcal{H})$ for Hilbert Space $\mathcal{H}$:

$$\Lambda_1(A) \supseteq \Lambda_2(A) \supseteq \Lambda_3(A) \supseteq \ldots \supseteq \Lambda_N(A)$$

- **Proof:**
  - If we can show:

$$\Lambda_k(A) \subseteq \Lambda_{k-1}(A)$$

    for arbitrary k, then we are done.
  - Let $\lambda \in \Lambda_k(A)$, where λ is complex. Suppose $\exists$ a rank-k projection P_1 onto Hilbert space $\mathcal{H}_1 \subseteq \mathcal{H}$ such that:

$$P_1 A P_1 = \lambda P_1 \qquad (*)$$

  - Let P_2 be a rank-(k-1) projection such that

$$P_1 P_2 = P_2 P_1 = P_2 \qquad \text{(From Property)}$$

Proof of Property 6 Ctd...

- **Proof Ctd:**
  - Apply P_2 to the left and right of (*):

$$\underbrace{P_2 P_1}_{=P_2} A \underbrace{P_1 P_2}_{=P_2} = P_2 \lambda \underbrace{P_1 P_2}_{=P_2}$$

$$P_2 A P_2 = P_2 \lambda P_2$$

$$P_2 A P_2 = \lambda P_2 P_2$$

$$P_2 A P_2 = \lambda (P_2)^2$$

$$P_2 A P_2 = \lambda P_2$$

$$\Rightarrow \lambda \in \Lambda_{k-1}(A)$$

$$\Rightarrow \Lambda_k \subseteq \Lambda_{k-1}(A)$$

Geometry of Hermitian Operators

- The eigenvalues of Hermitian operators are always real, so the higher-rank numerical range will form nested intervals on the real line.
- A general description is given by the following theorem:

Theorem ([1])

Let H be an $N \times N$ Hermitian operator with non-degenerate spectrum given by $Spec(H) = \{\lambda_1, \lambda_2, \ldots \lambda_N\}$ where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$. Let $k \in \mathbb{Z}^+$ be fixed with $1 \leq k \leq N$. The rank-k numerical range of H is:

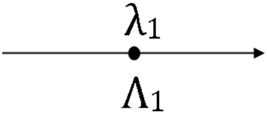
1. *A non-degenerate closed interval if $\lambda_k < \lambda_{N-k+1}$*
2. *A singleton set if $\lambda_k = \lambda_{N-k+1}$*
3. *An empty set if $\lambda_k > \lambda_{N-k+1}$*

Examples: Hermitian Operators

- This theorem is easier to see with some examples:

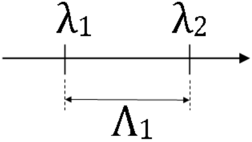
When $N = 1, k = \{1\}$

k	$N-k+1$
1	1



When $N = 2, k = \{1, 2\}$

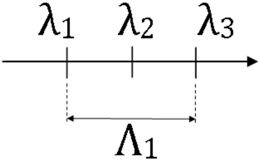
k	$N-k+1$
1	2
2	1



Examples of Theorem Ctd...

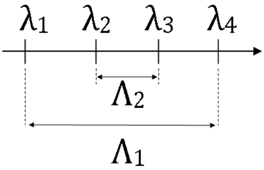
When $N = 3, k = \{1, 2, 3\}$

k	$N - k + 1$
1	3
2	2
3	3



When $N = 4, k = \{1, 2, 3, 4\}$

k	$N - k + 1$
1	4
2	3
3	2
4	1



Hermitian Operators Ctd...

- With this construction, the following proposition becomes evident:

Proposition ([1])

Let A be an NxN matrix and and suppose $2k \geq N$. Then the $rank-k$ numerical range $\Lambda_k(A)$ is an empty set or a singleton.

- This proposition is true for all types of operators.
- This proposition however does not guarantee non-emptiness.
- So an interesting question is under what conditions is $\Lambda_k(A)$ guaranteed to be non-empty?
- Answer:...to be revealed!

Normal Operators (Including Unitary Operators)

- These operators can have complex spectra.
- Choi et al introduced the following theorem regarding the convexity of the higher-rank numerical range.

Theorem ([1])

For a normal operator $T \in \mathcal{L}(\mathcal{H})$ of dimension N:

$$\Lambda_k(T) = \cap \ conv\ (\Gamma), \qquad \Gamma \subseteq \{Eigenvalues\ of\ T\}$$
$$|\Gamma| = N - k + 1$$

- Γ is the set of points in $\mathbb{C}$ that coincides with all $N - k + 1$ points of $Spec(T)$. This theorem presents an interesting way to construct $\Lambda_k(T)$ by examining the intersection of the convex hulls of point subsets.

Unitary and Normal Operators

- The following theorem proved by Li and Sze is actually another way of stating the previous theorem:

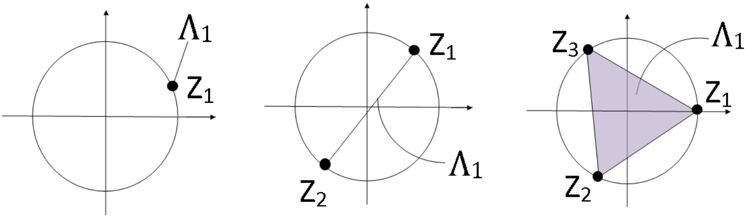
Theorem ([4])

For an NxN normal matrix T , $\Lambda_k(T)$ coincides with the intersection of convex hulls conv (Γ), where Γ is an $(N+1-k)-$point subset of the spectrum of T.

- This theorem shows that $\Lambda_k(T)$ is convex whenever $\Lambda_k(T)$ ia non-empty and T is normal.
- We will look at some examples to get a better idea.

Examples: Unitary Operators

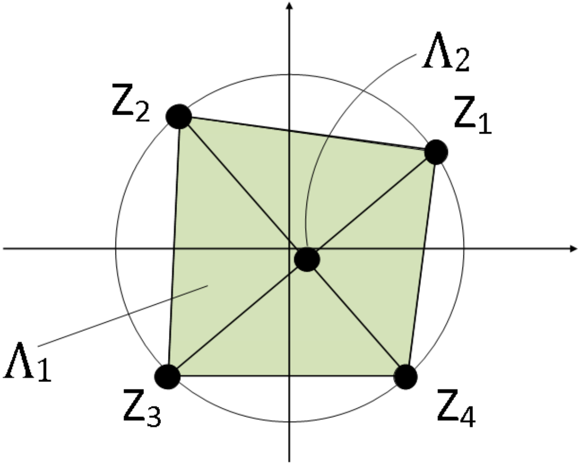
- We will begin with some examples of unitary operators.
- A very nice property of unitary operators is that their eigenvalues lie on the unit circle $\mathbb{D}$ in the complex plane.
- For the moment, we will assume non-degenerate spectra. The cases where N=1, N=2 and N=3 are not that interesting.



N=1 N=2 N=3

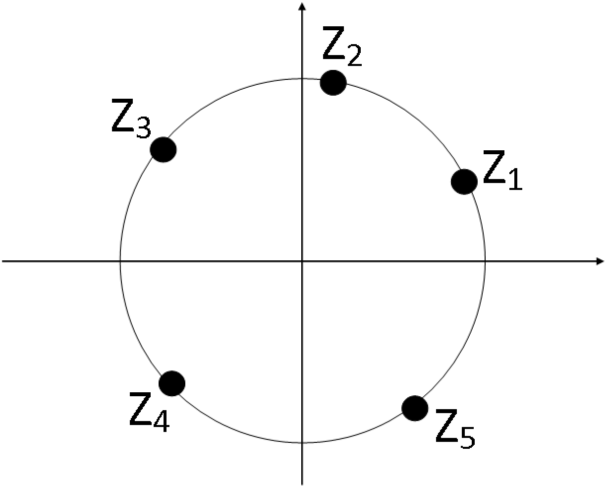
Example: N=4, k=1

- The case of N=4 is only slightly interesting:

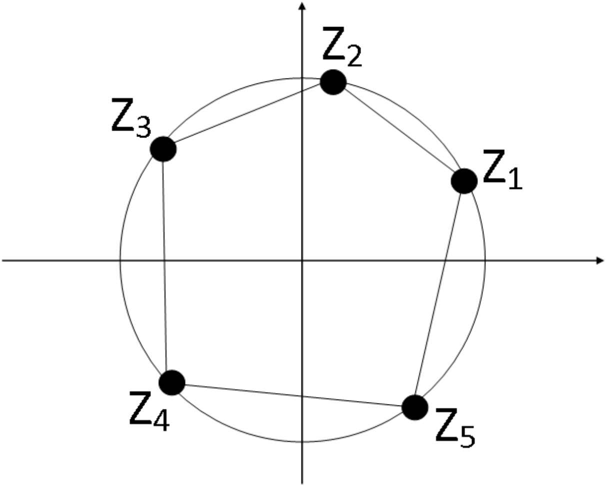


Example: N=5, k=1

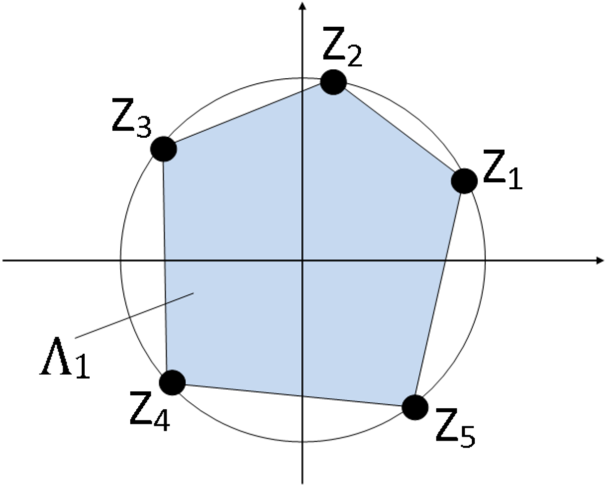
- Things start to get more interesting at N=5. We'll start by constructing $\Lambda_1(U)$ of a unitary operator $U \in \mathcal{L}(\mathbb{C}^5)$:



Example: N=5, k=1 Ctd...

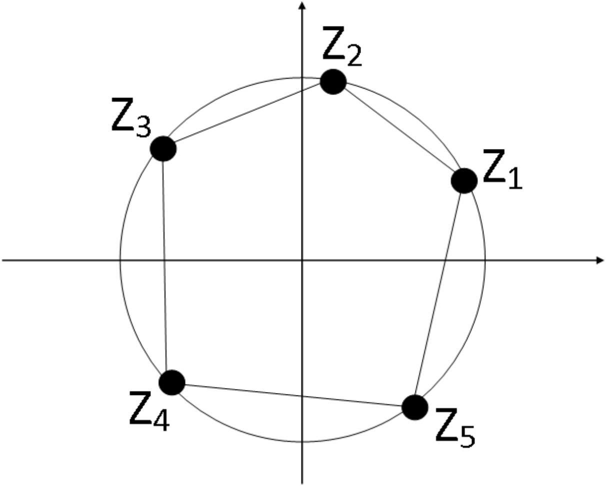


Example: N=5, k=1 Ctd...

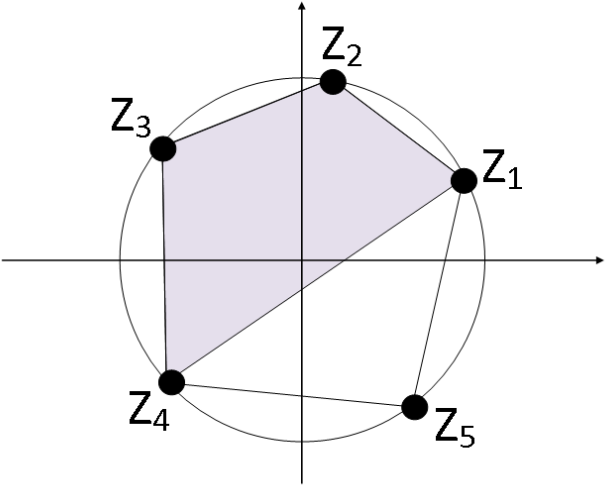


Example: N=5, k=2

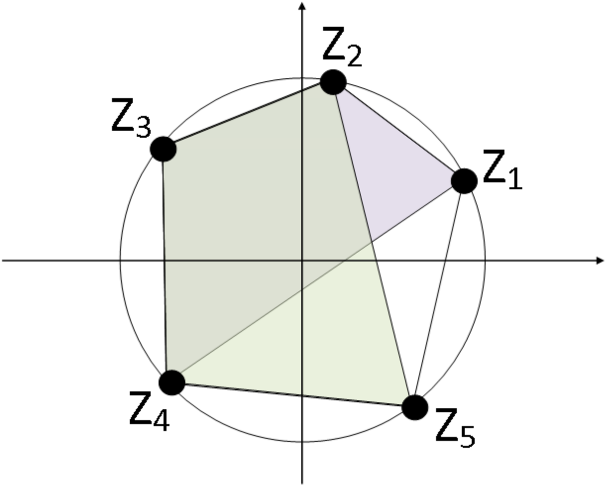
- Let's now construct $\Lambda_2(U)$. Since $k = 2$, $N - k + 1 = 4$, so we consider the intersection of the convex hulls of all 4-tupples.



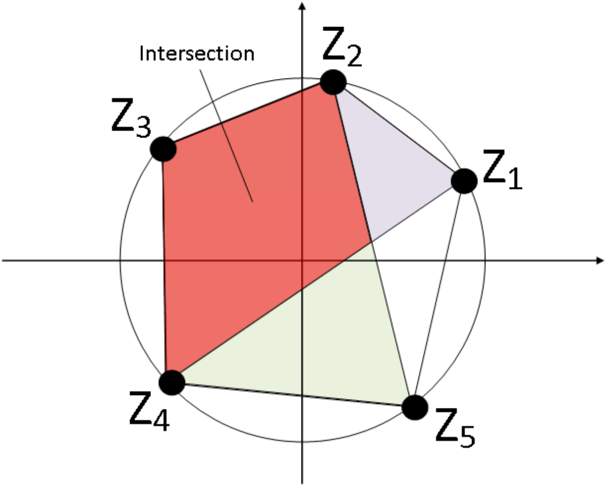
Example: N=5, k=2 Ctd...



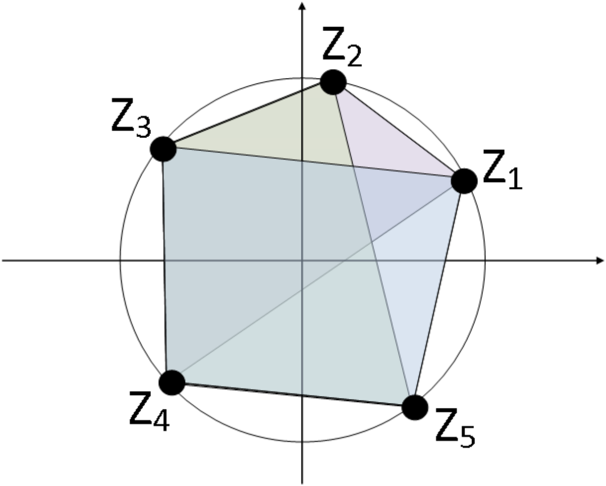
Example: N=5, k=2 Ctd...



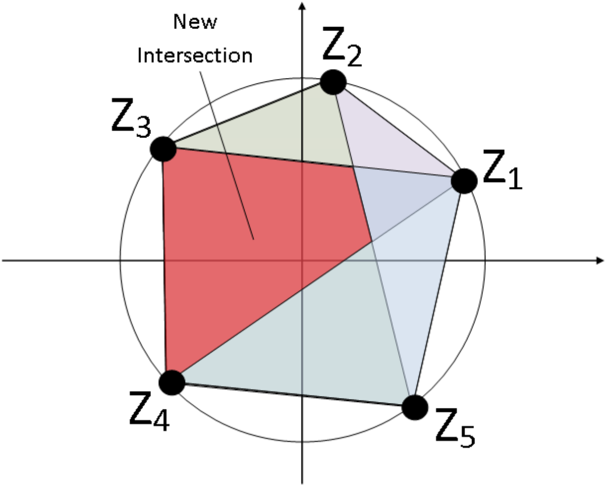
Example: N=5, k=2 Ctd...



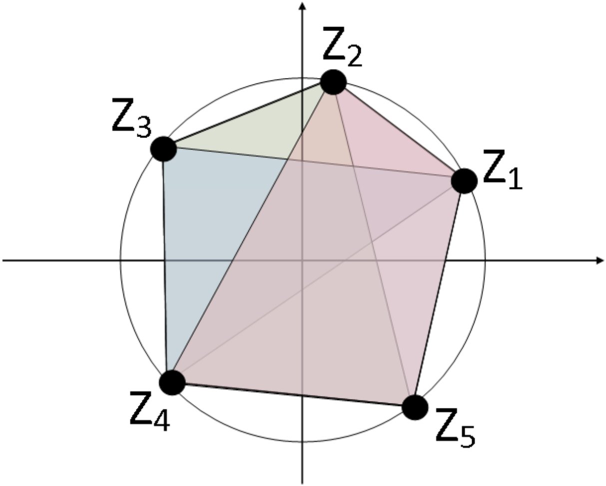
Example: N=5, k=2 Ctd...



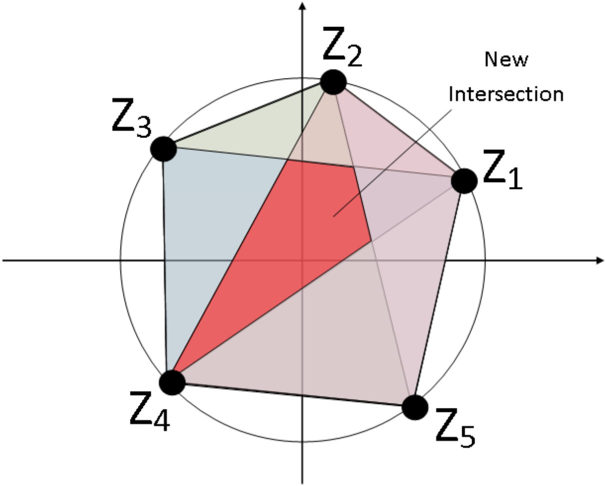
Example: N=5, k=2 Ctd...



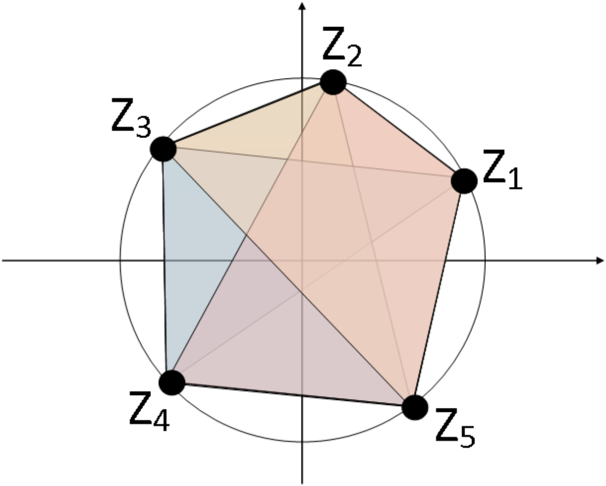
Example: N=5, k=2 Ctd...



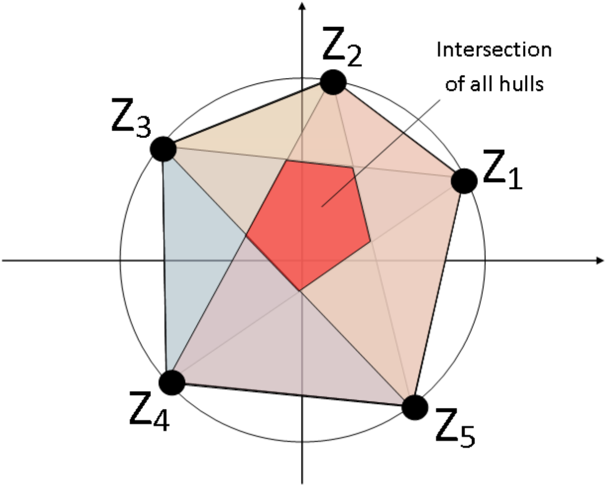
Example: N=5, k=2 Ctd...



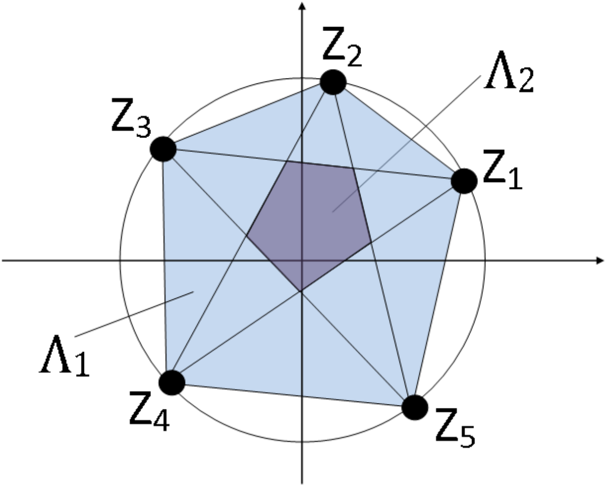
Example: N=5, k=2 Ctd...



Example: N=5, k=2 Ctd...

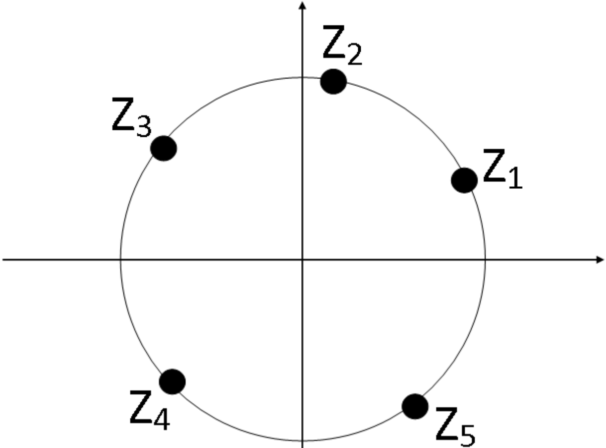


Example: N=5, k=2 Ctd...

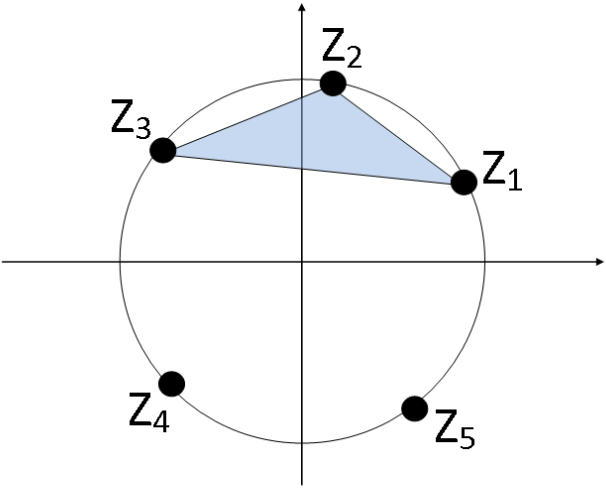


Example: N=5, k=3

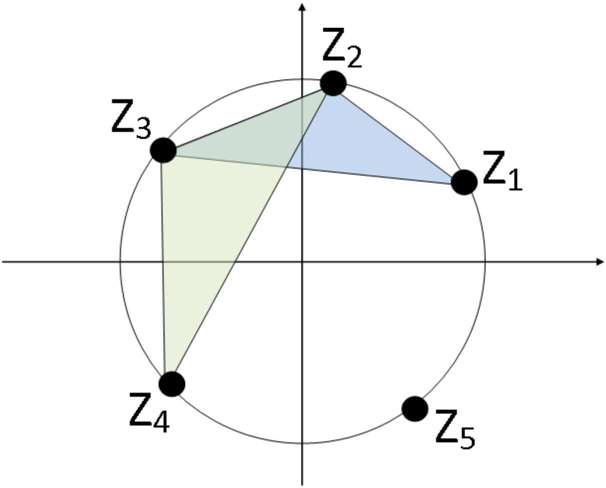
- We will see for N=5, the rank-3 numerical range is an empty set. To do this geometrically, we consider the convex hulls of all triples $(N - k + 1 = 3)$:



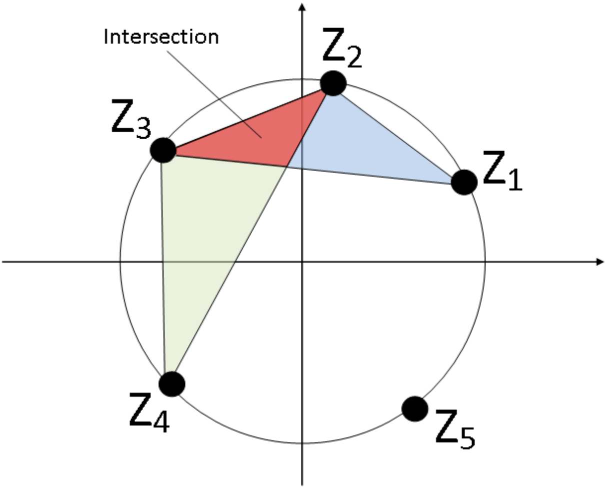
Example: N=5, k=3 Ctd...



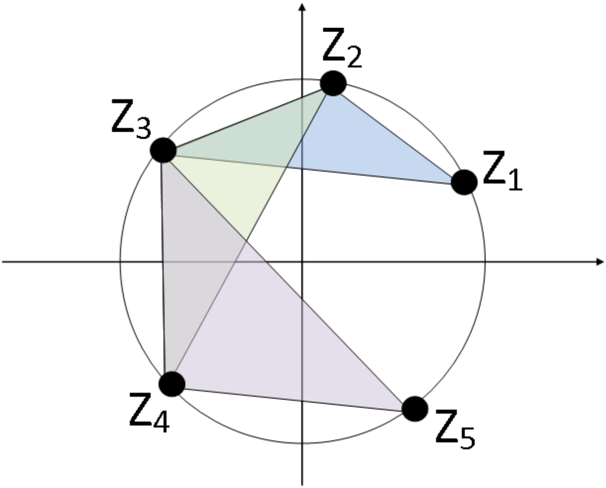
Example: N=5, k=3 Ctd...



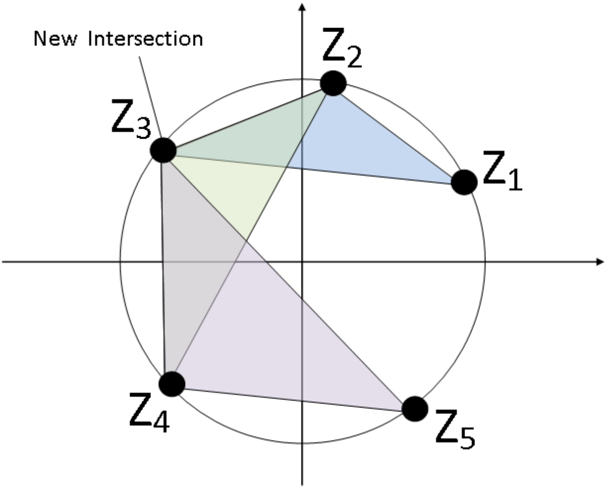
Example: N=5, k=3 Ctd...



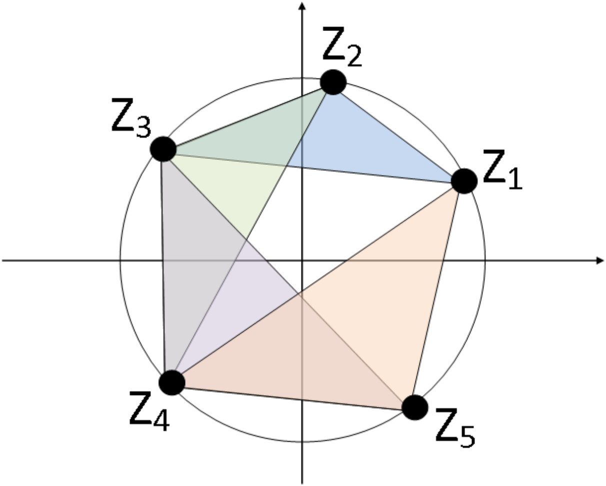
Example: N=5, k=3 Ctd...



Example: N=5, k=3 Ctd...



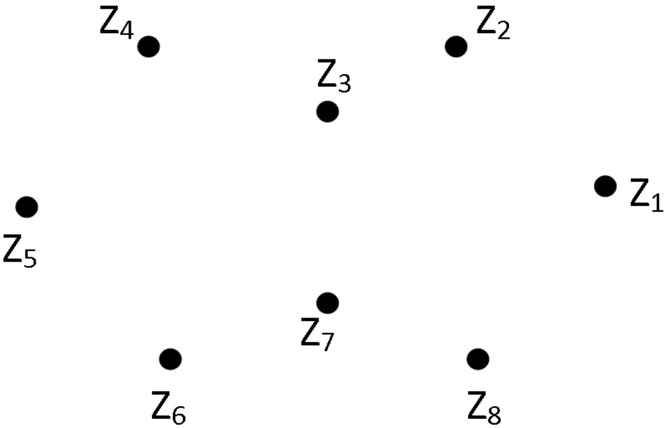
Example: N=5, k=3 Ctd...



- We already see that there is no common intersection of these 4 triples.

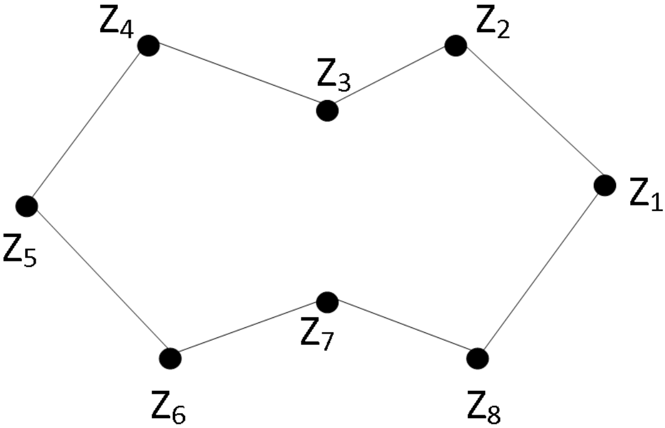
Examples: Normal Operators

- We will move onto the more general case of normal operators.
- The spatial arrangement in $\mathbb{C}$ is not fixed, so we may end up with a figure such as the one below for some normal T:



Example: Normal Operators Ctd...

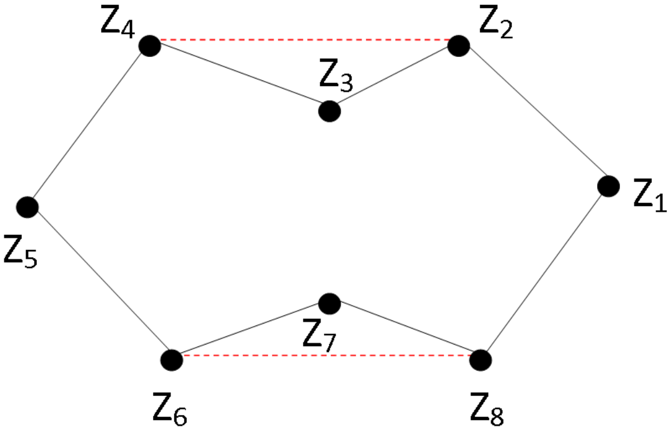
- Let's construct $\Lambda_1(T)$ in the usual way:



- Does anyone see the problem here?

Example: Normal Operators Ctd...

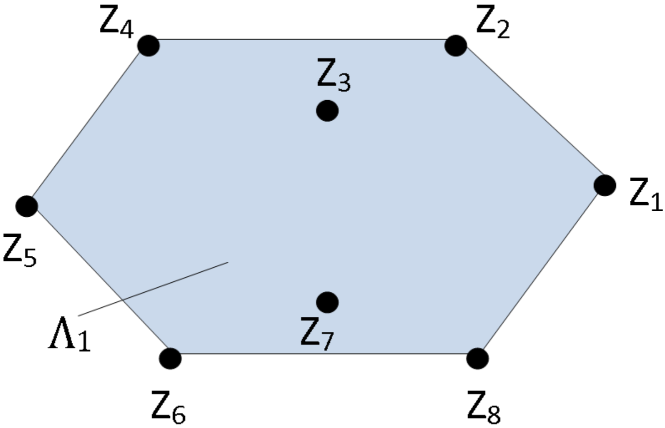
- This picture of $\Lambda_1(T)$ is not convex, so let's make it convex:



- Constructing $\Lambda_1(T)$ in this manner gives us a convex hull.

Example: Normal Operators Ctd...

- So $\Lambda_1(T)$ is:



- Notice that there are eigenvalues in the interior of $\Lambda_1(T)$.

Discussion and New Results

Discussion and New Results

Discussion

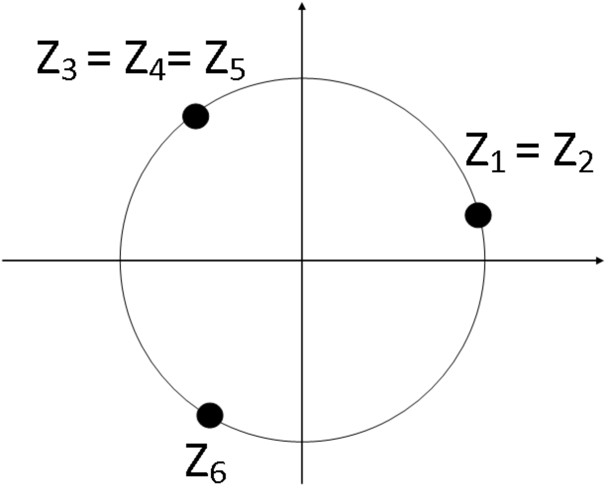
- What happens to the geometry of of the higher rank numerical range when we have degenerate spectra?
- What interesting conclusions about the geometry of the higher rank numerical range can we arrive at?
- Given N and k, when are we guaranteed that $\Lambda_k(A)$ is non-empty for some operator A?
- Are there any bounds on the number of sides of the convex polygon of $\Lambda_k(A)$.
- Are there any other interesting ways to construct $\Lambda_k(A)$?

Degeneracy

- If we get a Hermitian operator with a degenerate spectrum, then some of the nested intervals will not be proper subsets but equal sets.
- In the unitary case, the number of sides of the convex polygon decreases. We may also have *dimension reduction*.
- For instance, consider a unitary $U \in \mathcal{L}(\mathbb{C}^6)$, and lets say $Z_1 = Z_2$, $Z_3 = Z_4 = Z_5$ and Z_6 is unique.

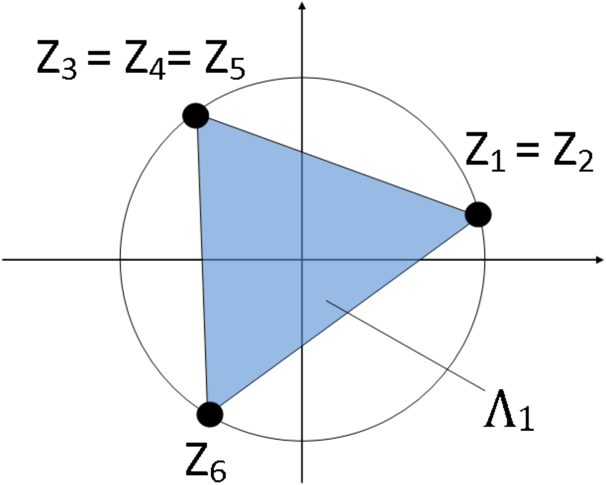
Degeneracy Ctd...

- Suppose their spatial arrangement in $\mathbb{C}$ is:



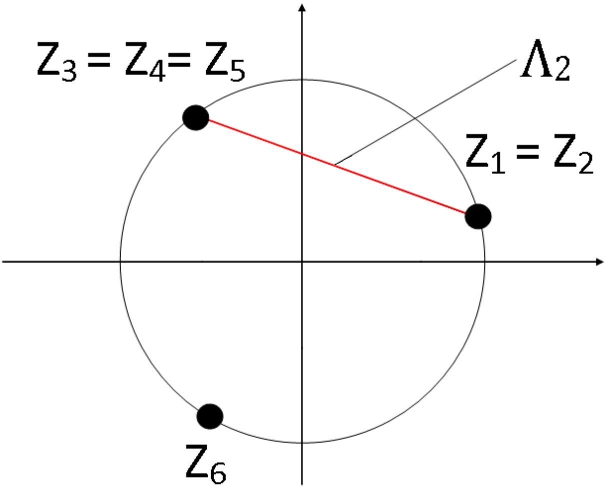
Degeneracy Ctd...

- $\Lambda_1(U)$ is a hull with 3 sides



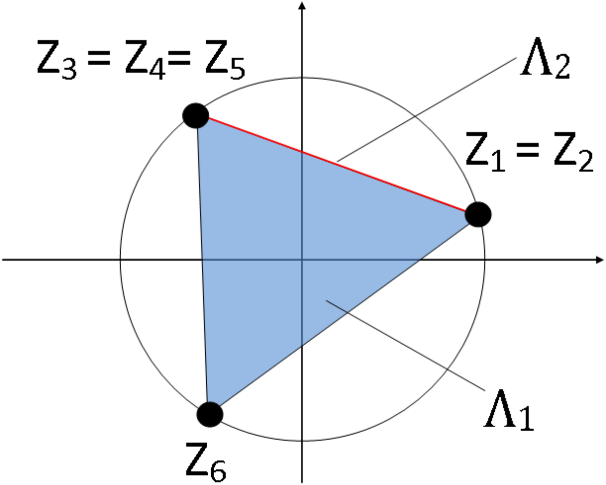
Degeneracy Ctd...

- $\Lambda_2(U)$ is a line:



Degeneracy Ctd...

- $\Lambda_3(U)$ is empty:



Non-emptiness

- Choi, Kribs, Holbrook and Życkowski presented the following theorem regarding non-emptiness for the unitary case:

Theorem ([3])

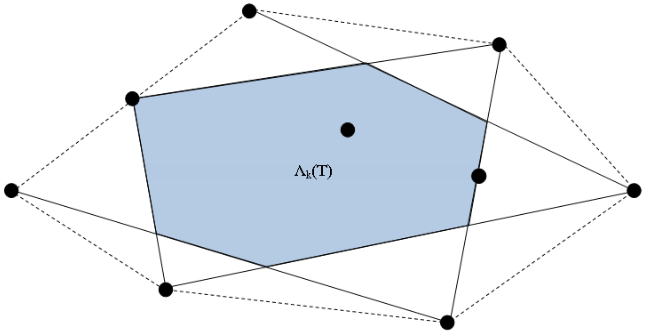
Let unitary $U \in \mathcal{L}(\mathbb{C}^N)$ with non-degenerate spectrum and let $k \geq 1$. Then the following are true for $\Lambda_k(U)$:

1. *If $2k > N$, then $\Lambda_k(U) = \emptyset$.*
2. *If $2k = N$, then $\Lambda_k(U)$ is empty if the line segments $[z_j, z_{j+1}]$ do not intersect. If they do intersect, then we get a singleton set.*
3. *If $2k < N < 3k - 2$, then $\Lambda_k(U)$ can be either or non-empty.*
4. *If $3k - 2 \leq N$, then $\Lambda_k(U)$ is* **always** *non-empty, whether the eigenvalues are distinct or not.*

Bounds on the Convex Polygon

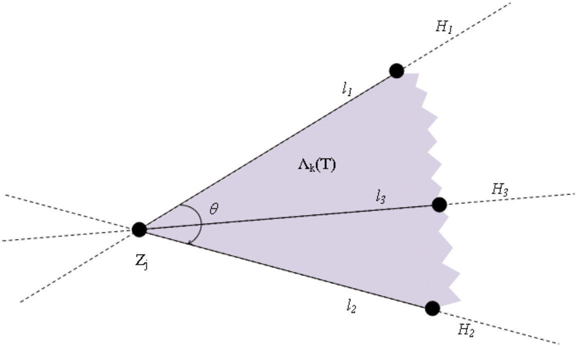
- Gau, Li, Poon, and Sze showed that the rank-k numerical range of a normal matrix with N distinct eigenvalues is a polygon with at most N sides [5].

- The presenter also proved it in an entirely different way:



Bounds on the Convex Polygon Ctd...

- Sketch of (my) proof:



- We looked at the intersection of half-planes and used the axioms of Euclidean geometry and a counting argument.

Bounds on the Convex Polygon Ctd...

- A nice result which we obtained while proving this result:

Proposition

Let unitary $U \in \mathcal{L}(\mathbb{C}^N)$ with non-degenerate spectrum given by $Spec(U) = \{z_1, z_2, \ldots z_N\}$ such that $arg(z_j) = \theta_j$ and $0 \leq \theta_1 < \theta_2 < \ldots < \theta_N < 2\pi$ are arranged counterclockwise on $\mathbb{D}$. Any line segment of the boundary of $\Lambda_k(U)$ must lie on one of the line segments $\overline{z_j z_{j+k}}$ for some integer j between 1 and N, provided that $3k - 2 \leq N$.

(Note that addition is performed modulo N in the subscript).

Interesting Ways to Construct $\Lambda_k(U)$

- We already saw how to construct of $\Lambda_k(U)$ by intersecting the convex hulls of $N-k+1$ point subsets for a unitary U.
- Now suppose $\exists q \in \mathbb{Z}^+$ such that $N = q \cdot k$, then $\Lambda_k(U)$ can be constructed by intersecting k polygons, each with q sides.
- We can associate each polygon to a permutation group; there will be k groups with q elements in each group:

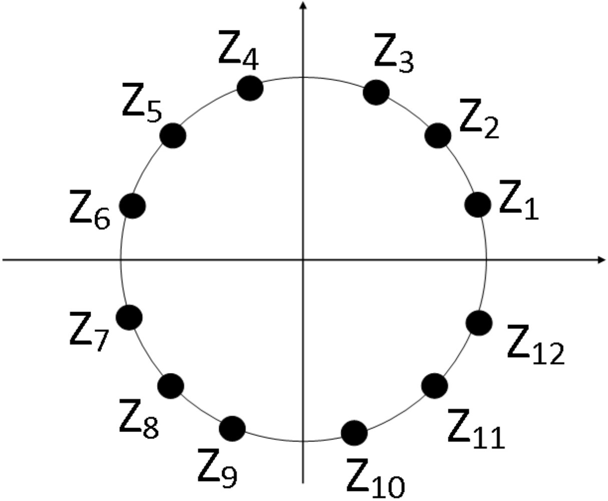
$$\text{k groups} \left\{ \underbrace{\begin{array}{cccc} (1, & 1+k, & \dots & 1+(q-1)k) \\ (2, & 2+k, & \dots & 2+(q-1)k) \\ \vdots & \vdots & \vdots & \vdots \\ (k, & 2k, & \dots & qk) \end{array}}_{\text{q elements}} \right.$$

Example: N=12, k=3

- Example: N=12, k=3.
- $12 = 4 \cdot 3$, so there will be 3 groups with 4 elements in each group:
  1. (1, 4, 7, 10)
  2. (2, 5, 8, 11)
  3. (3, 6, 9, 12)
- Let's construct $\Lambda_3(U)$ by looking at the intersection of these 3 polygons.

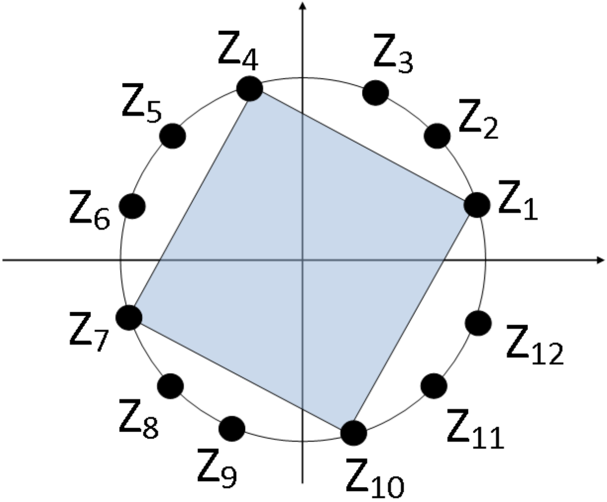
Example: N=12, k=3

- Assume non-degenerate spectra for some unitary $U \in \mathcal{L}(\mathbb{C}^{12})$:



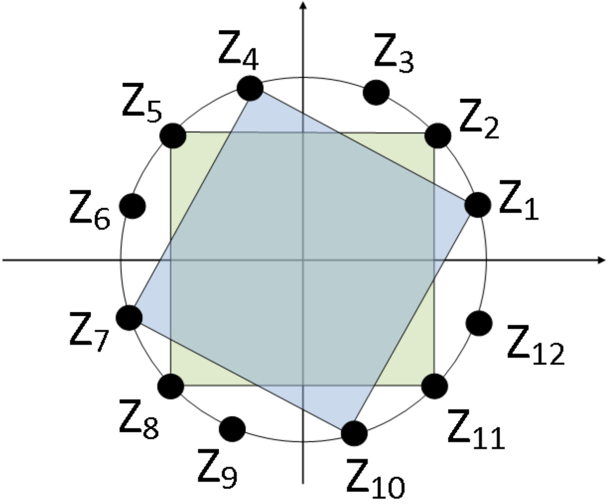
Example: N=12, k=3 Ctd...

- (1, 4, 7, 10)



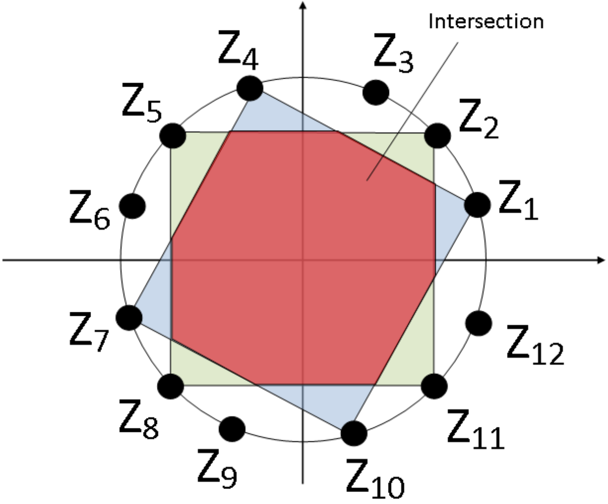
Example: N=12, k=3 Ctd...

- (1, 4, 7, 10), (2, 5, 8, 11)



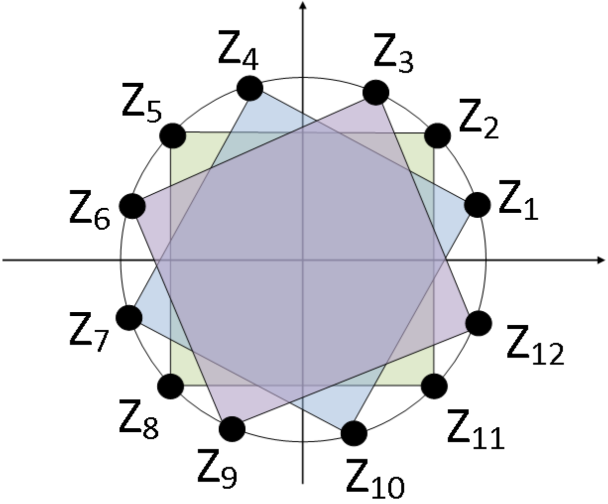
Example: N=12, k=3 Ctd...

- Intersection of (1, 4, 7, 10), (2, 5, 8, 11)



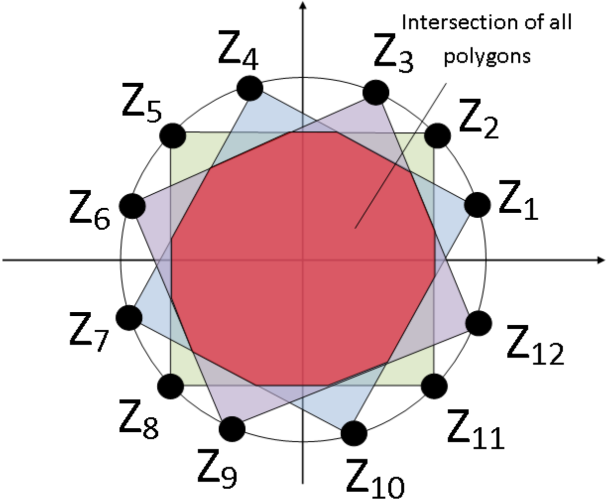
Example: N=12, k=3 Ctd...

- (1, 4, 7, 10), (2, 5, 8, 11), (3, 6, 9, 12)



Example: N=12, k=3 Ctd...

- Intersection of (1, 4, 7, 10), (2, 5, 8, 11), (3, 6, 9, 12)



Example: N=12, k=4

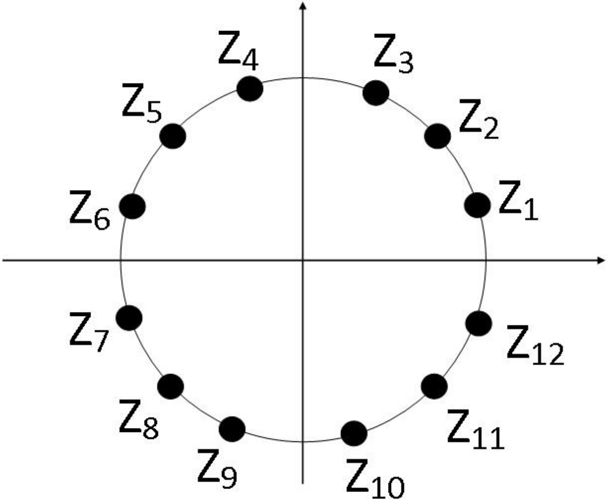
- Example: N=12, k=4.
- $12 = 3 \cdot 4$, so there will be 4 groups with 3 elements in each group:

  1. (1, 5, 9)
  2. (2, 6, 10)
  3. (3, 7, 11)
  4. (4, 8, 12)

- Let's construct $\Lambda_4(U)$ by looking at the intersection of these 4 polygons.

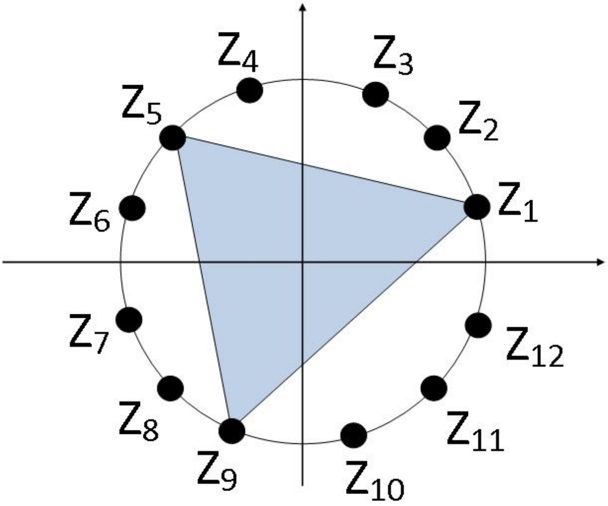
Example: N=12, k=4

- Assume non-degenerate spectra for some unitary $U \in \mathcal{L}(\mathbb{C}^{12})$:



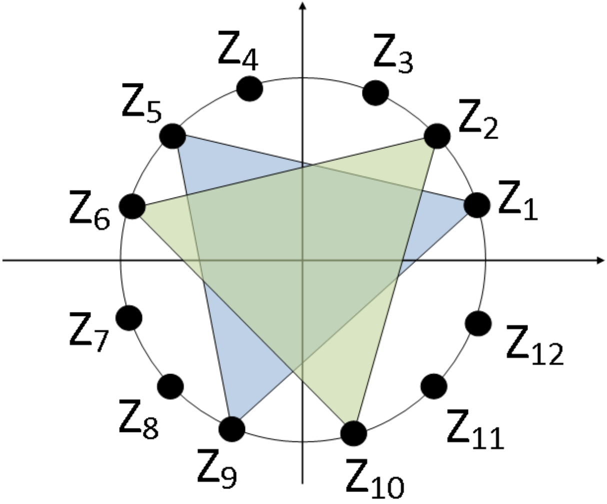
Example: N=12, k=4 Ctd...

- (1, 5, 9)



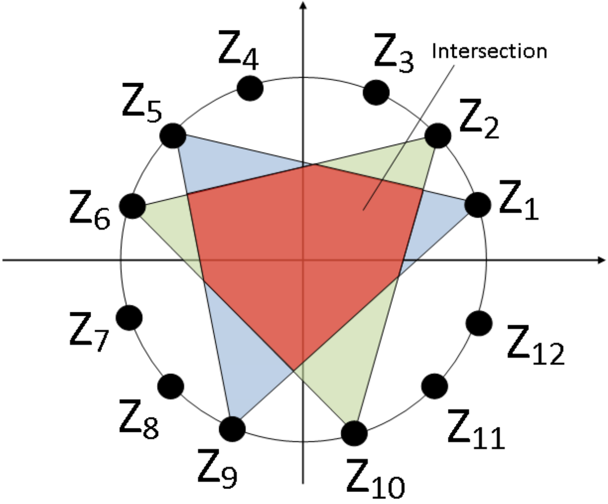
Example: N=12, k=4 Ctd...

- (1, 5, 9), (2, 6, 10)



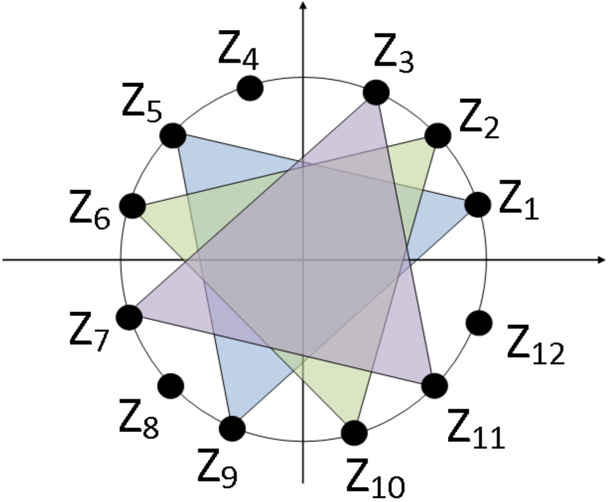
Example: N=12, k=4 Ctd...

- intersection of (1, 5, 9), (2, 6, 10)



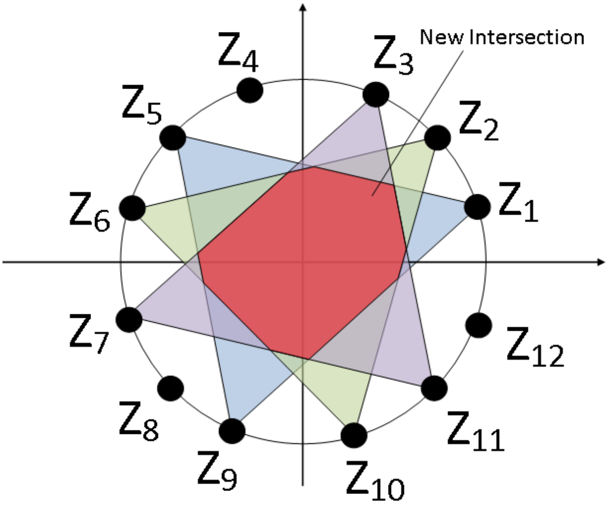
Example: N=12, k=4 Ctd...

- (1, 5, 9), (2, 6, 10), (3, 7, 11)



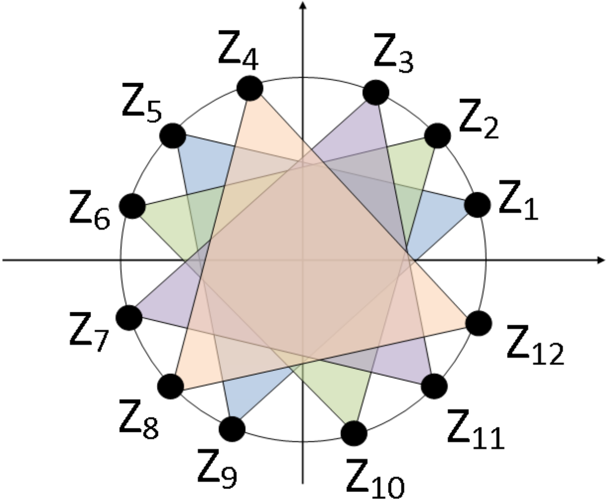
Example: N=12, k=4 Ctd...

- Intersection of (1, 5, 9), (2, 6, 10), (3, 7, 11)



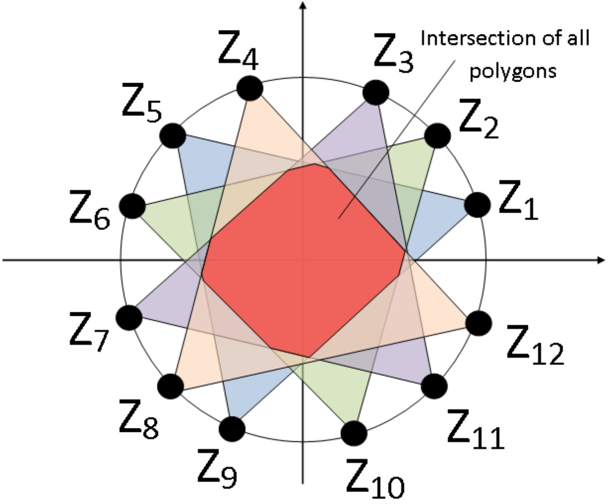
Example: N=12, k=4 Ctd...

- (1, 5, 9), (2, 6, 10), (3, 7, 11), (4, 8, 12)



Example: N=12, k=4 Ctd...

- Intersection of (1, 5, 9), (2, 6, 10), (3, 7, 11), (4, 8, 12)



Interesting Ways to Construct $\Lambda_k(U)$ Ctd...

- Suppose $\nexists q \in \mathbb{Z}^+$ such that $N = q \cdot k$, then $\Lambda_k(U)$ can be constructed by drawing a continuous succession of N line segments between eigenvalues and looking at the intersection of the half-planes formed by these line segments.
- The associated permutation group of this construction will have N elements:

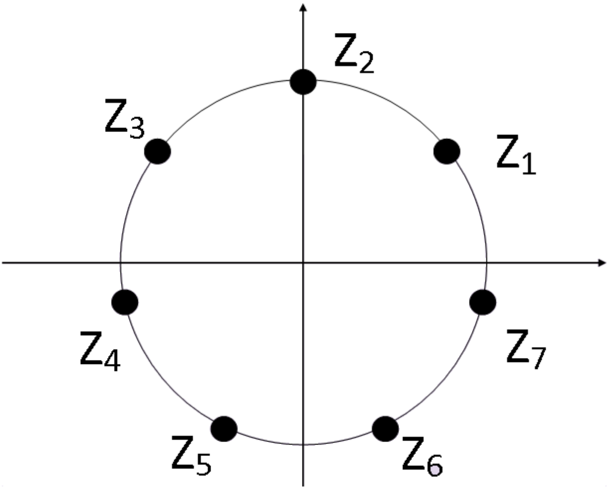
$$\underbrace{(1, 1 + k, \ldots 1 + (N - 1)k)}_{\text{N elements}}$$

- A consequence from elementary number theory is that we can construct $\Lambda_k(U)$ this way whenever N is prime.
- Example: $N = 7$, $k = 2$. The associated permutation group is:

$$(1,\ 3,\ 5,\ 7,\ 2,\ 4,\ 6)$$

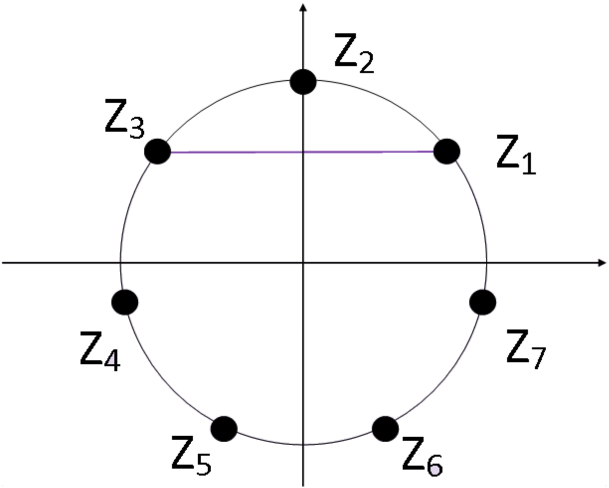
Example: N=7, k=2

- Assume non-degenerate spectra for some unitary $U \in \mathcal{L}(\mathbb{C}^7)$:



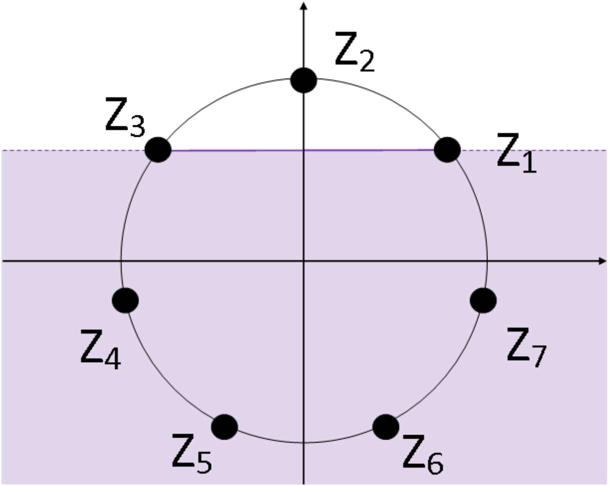
Example: N=7, k=2 Ctd...

- (**1, 3** , 5, 7, 2, 4, 6)



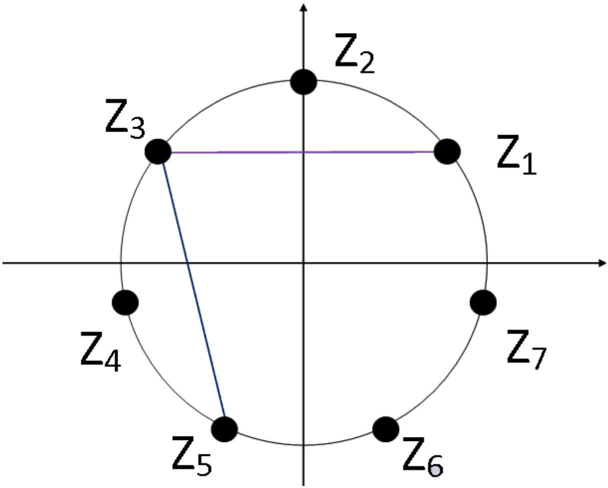
Example: N=7, k=2 Ctd...

- (**1, 3** , 5, 7, 2, 4, 6)



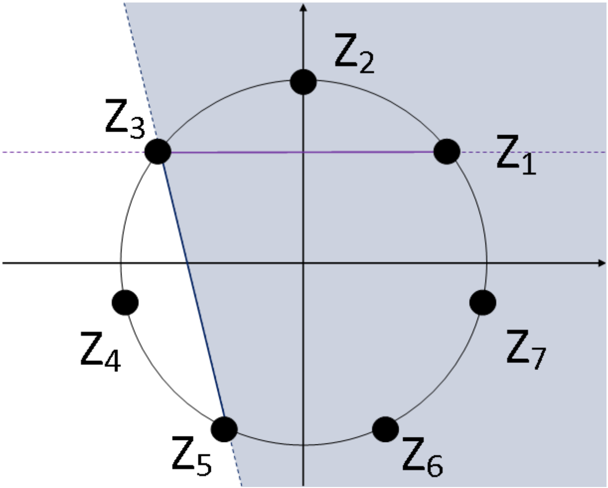
Example: N=7, k=2 Ctd...

- (1, **3, 5** , 7, 2, 4, 6)



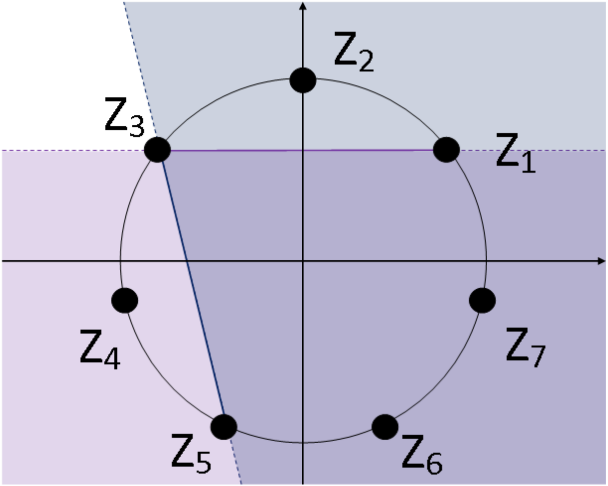
Example: N=7, k=2 Ctd...

- (1, **3, 5** , 7, 2, 4, 6)



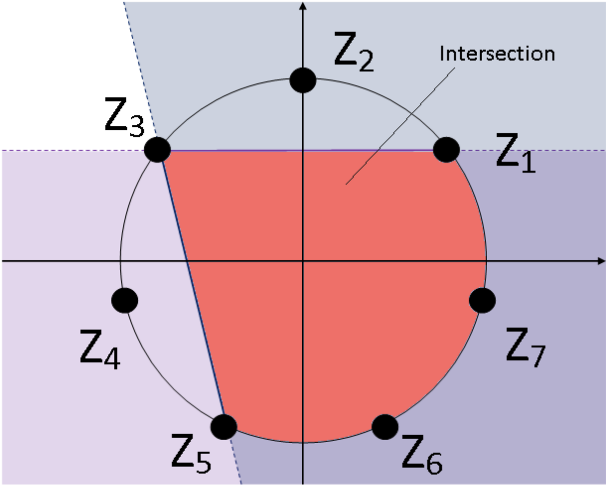
Example: N=7, k=2 Ctd...

- (1, **3, 5** , 7, 2, 4, 6)



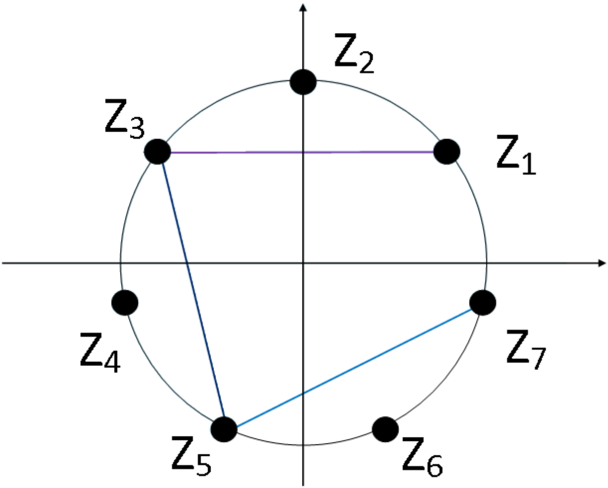
Example: N=7, k=2 Ctd...

- (1, **3, 5** , 7, 2, 4, 6)



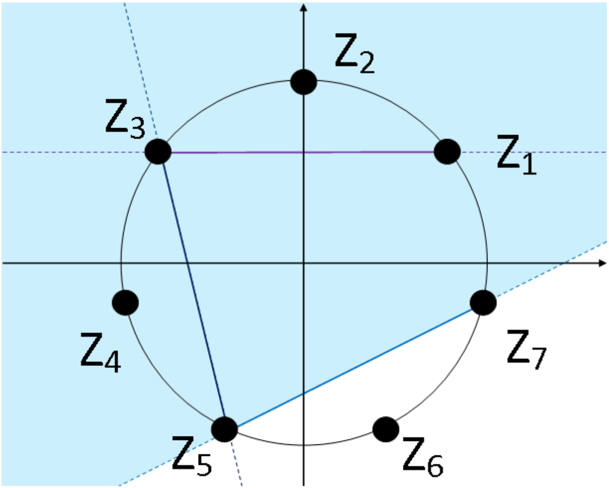
Example: N=7, k=2 Ctd...

- (1, 3, **5, 7** , 2, 4, 6)



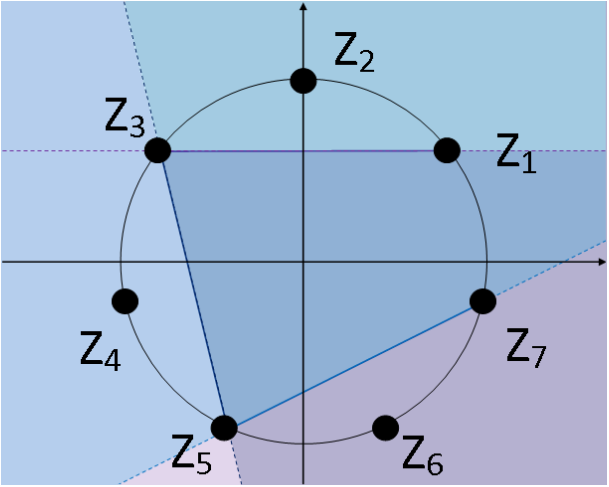
Example: N=7, k=2 Ctd...

- (1, 3, **5, 7** , 2, 4, 6)



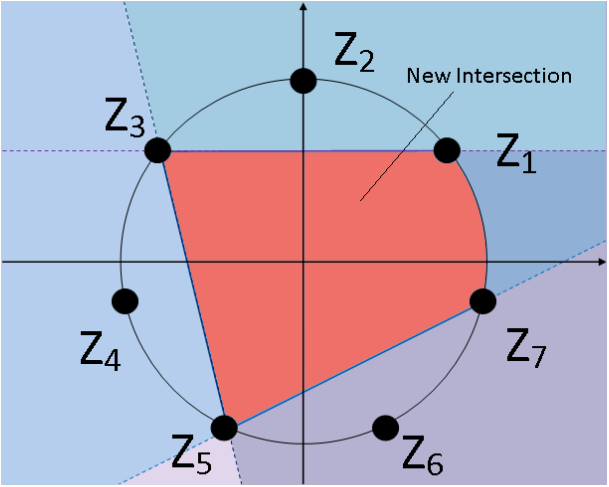
Example: N=7, k=2 Ctd...

- (1, 3, **5, 7** , 2, 4, 6)



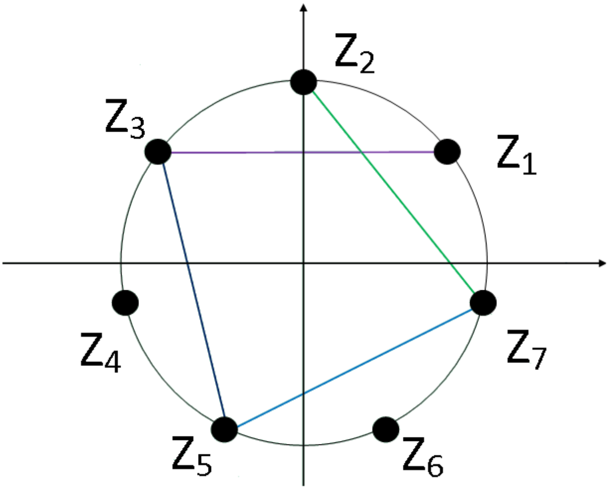
Example: N=7, k=2 Ctd...

- (1, 3, **5, 7** , 2, 4, 6)



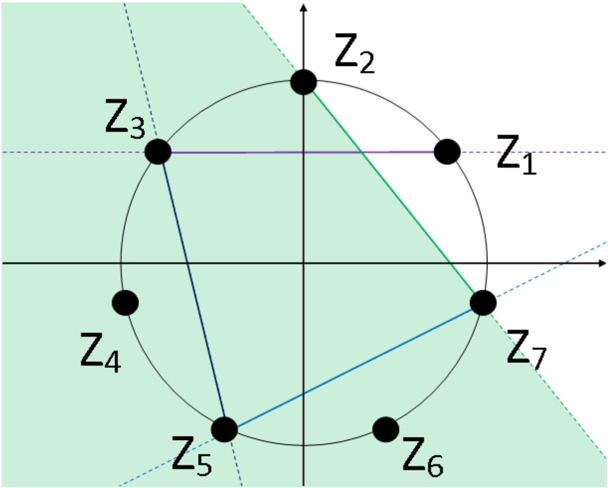
Example: N=7, k=2 Ctd...

- (1, 3, 5, **7, 2** , 4, 6)



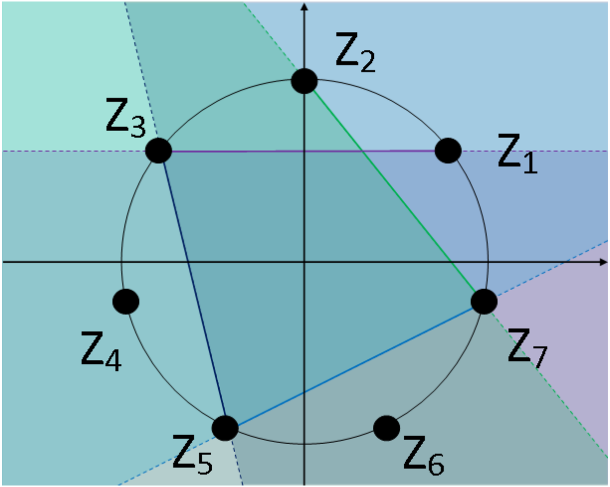
Example: N=7, k=2 Ctd...

- (1, 3, 5, **7, 2** , 4, 6)



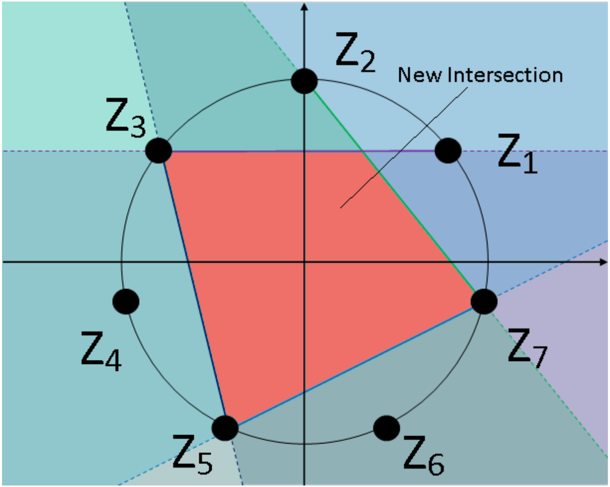
Example: N=7, k=2 Ctd...

- (1, 3, 5, **7, 2** , 4, 6)



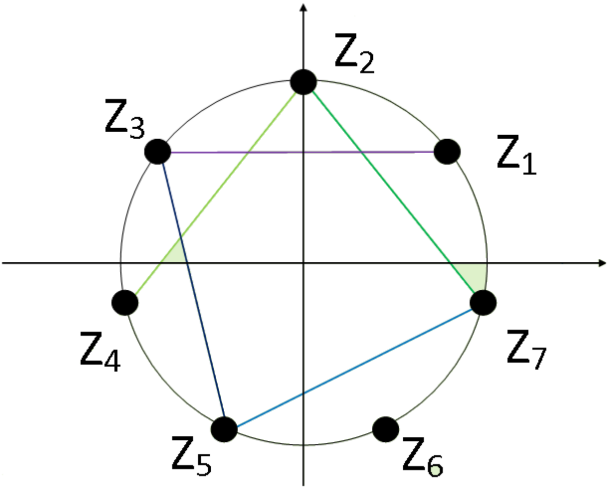
Example: N=7, k=2 Ctd...

- (1, 3, 5, **7, 2** , 4, 6)



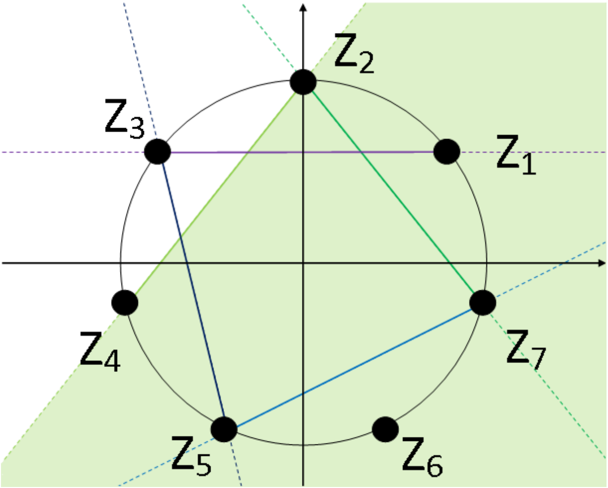
Example: N=7, k=2 Ctd...

- (1, 3, 5, 7, **2, 4** , 6)



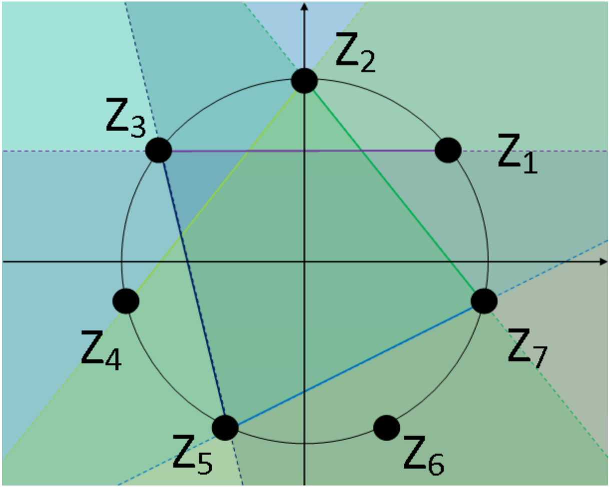
Example: N=7, k=2 Ctd...

- (1, 3, 5, 7, **2, 4** , 6)



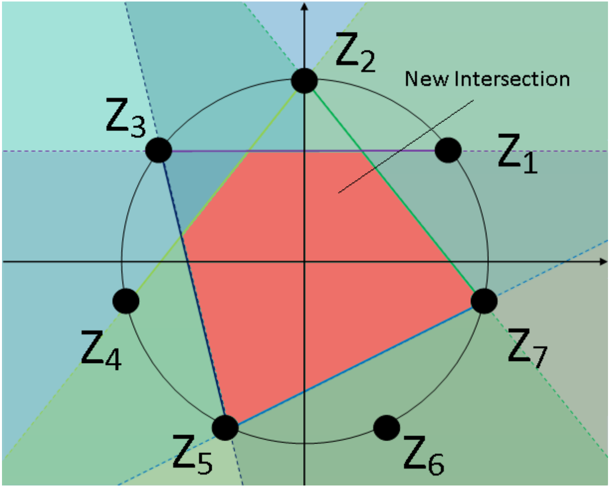
Example: N=7, k=2 Ctd...

- (1, 3, 5, 7, **2, 4** , 6)



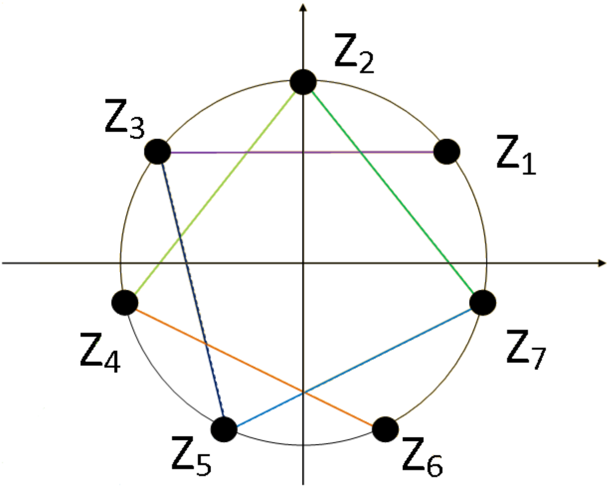
Example: N=7, k=2 Ctd...

- (1, 3, 5, 7, **2, 4** , 6)



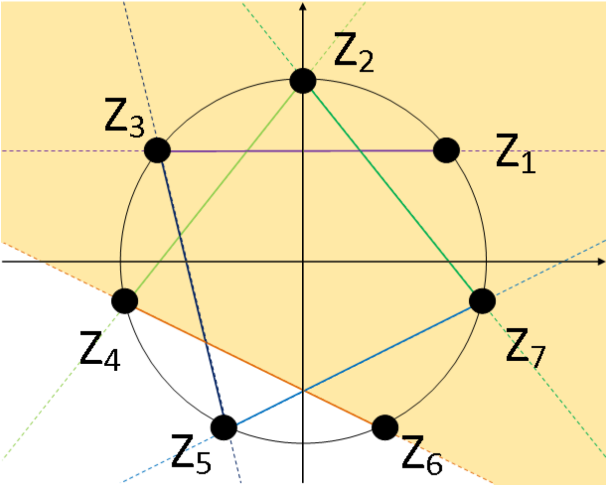
Example: N=7, k=2 Ctd...

- (1, 3, 5, 7, 2, **4, 6**)



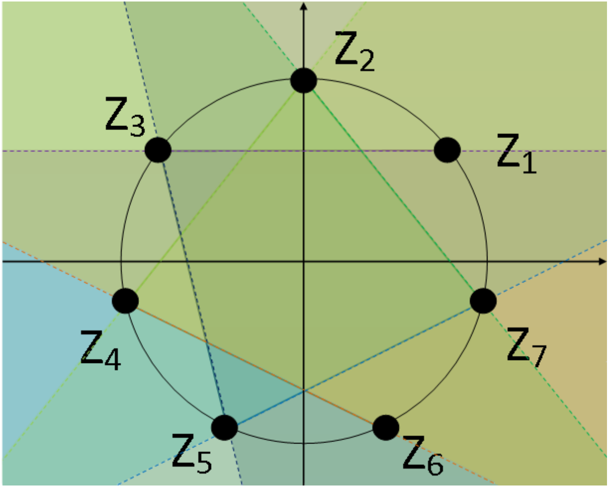
Example: N=7, k=2 Ctd...

- (1, 3, 5, 7, 2, **4, 6**)



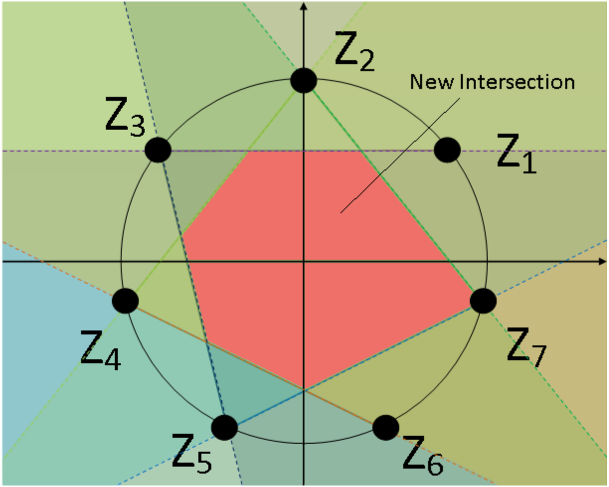
Example: N=7, k=2 Ctd...

- (1, 3, 5, 7, 2, **4, 6**)



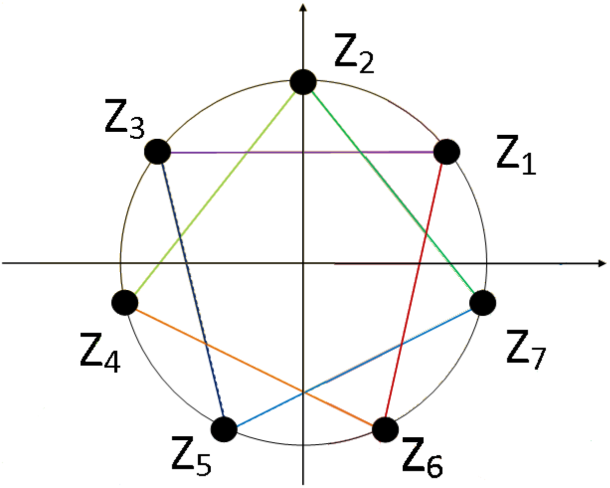
Example: N=7, k=2 Ctd...

- (1, 3, 5, 7, 2, **4, 6**)



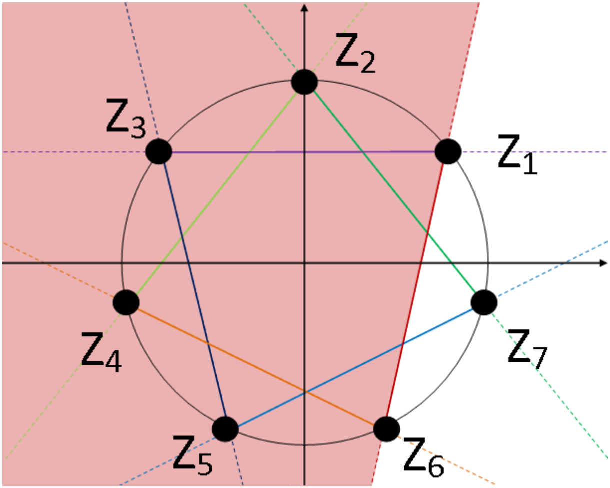
Example: N=7, k=2 Ctd...

- (**1** , 3, 5, 7, 2, 4, **6**)



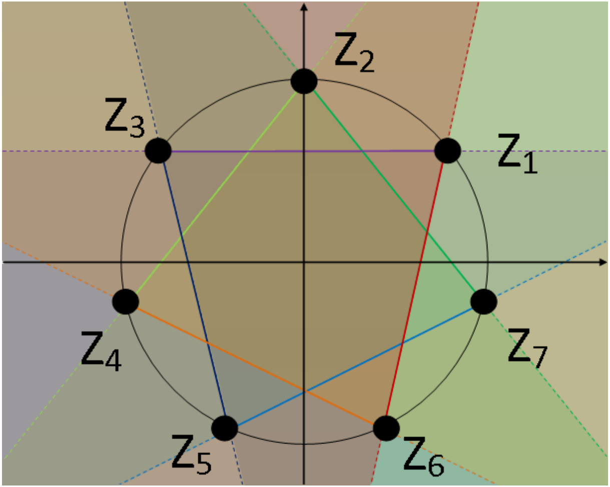
Example: N=7, k=2 Ctd...

- (**1** , 3, 5, 7, 2, 4, **6**)



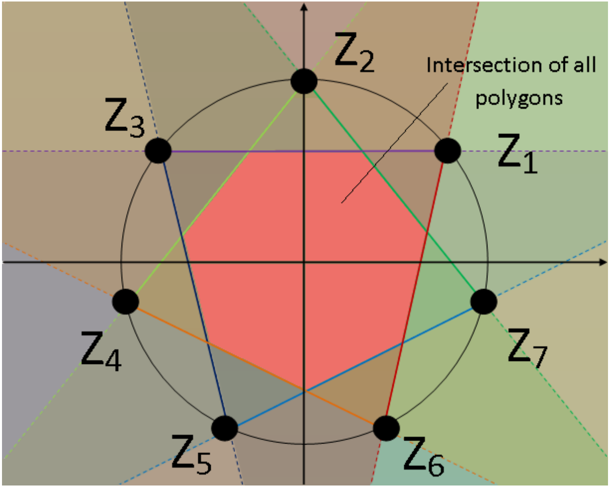
Example: N=7, k=2 Ctd...

- (**1** , 3, 5, 7, 2, 4, **6**)



Example: N=7, k=2 Ctd...

- (**1** , 3, 5, 7, 2, 4, **6**)



Applications in Quantum Data Error Correction

Applications in Quantum Data Error Correction

Introduction to Quantum Data Error Correction

- Quantum data error correction is a research area that's still in its infancy. It was only recently that researchers realized that data correction was possible (at least theoretically) with qubits.

- The reason we are interested in quantum data error correction is because several things can go wrong when transmitting qubits along a quantum channel:

  1. Bit flip errors
     $\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|1\rangle + \beta|0\rangle$
  2. Phase flip errors
     $\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|0\rangle - \beta|1\rangle$
  3. Both bit flip and phase flip errors
     $\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|1\rangle - \beta|0\rangle$

Introduction to Quantum Data Error Correction Ctd...

- Bit flip errors can be corrected are by applying the Pauli-X operator σ_x.
- Phase flip errors can be corrected are by applying the Pauli-Z operator σ_z.
- Some error correction techniques that work (in theory) are:
  1. The 3-qubit code [3]
  2. The 5-qubit code [3]
  3. Shor's 9-qubit code [3]
  4. The concatenated code [3]
- We will not discuss there techniques, but mention them so that the reader is aware that some techniques for quantum data error correction already exist.

Introduction to Quantum Data Error Correction Ctd...

- The way that they work is that *ancilla* qubits (extra qubits) are added to each $|0\rangle$ and $|1\rangle$ to get the encoded qubits $|0_{enc}\rangle$ and $|1_{enc}\rangle$.
- Then a *Von Neumann* measurement will detect if a phase-flip has occurred or if a bit-flip occurred on $|0_{enc}\rangle$ or $|1_{enc}\rangle$.
- Since ancilla bits are added, the error correcting technique will determine which bit has flipped. This is called the *error syndrome*.
- After that the appropriate recovery operation is performed to recover the original state.

The Non-trivial Nature of Quantum Data Error

- Using *classical* methods for correcting encoded bits fails in a quantum setting for several main reasons:

1. Classical error correction can not correct phase flips.
2. The *No-Cloning Theorem*:

Theorem (The No-Cloning Theorem)

There does not exist a superoperator O_S which can perform

$$|\psi\rangle\langle\psi| \otimes |\psi'\rangle\langle\psi'| \xrightarrow{O_S} |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|$$

where $|\psi'\rangle$ is a fixed state of ancilla (extra) bits.

This means that given a fixed state $|\psi'\rangle$ there does not exist any unitary operator U that can encode a super-position state as $U|\psi\rangle|\psi'\rangle = |\psi\rangle|\psi\rangle$

The Non-trivial Nature of Quantum Data Error Ctd...

3. The *Heisenberg Uncertainty* Principle:

Principle (Heisenberg Uncertainty)

$$\Delta x \Delta p_x \leq \frac{\hbar}{2}$$

where:

Δx *- is the uncertainty in the x-coordinate of the particle (position)*

Δp_x *- the uncertainty in the x-component of the particle's momentum*

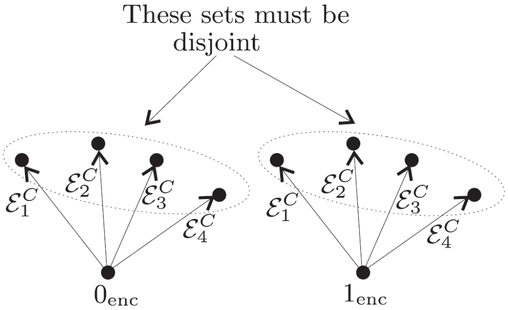
$\hbar$ *- The reduced Planck constant,* $6.626068 \times 10^{-34}/2\pi \; m^2 kg/s$

The Non-trivial Nature of Quantum Data Error Ctd...

- For us, it suffices to interpret item 3 as stating that any precise measurement on the atomic scale will cause some disturbance in the state of a quantum system.
- The (accurate) measurement of qubits in a quantum channel will cause a disturbance which will upset the integrity of the quantum system.
- Example: Think of trying to study a drop of ink or a coloured dye-crystal diffusing in a container of water using a microscope.

Correctable Code

- Quantum data error correction relies on being able to distinguish the different errors associated with an encoded $|0\rangle$ and $|1\rangle$ (We shall call them $|0_{enc}\rangle$ and $|1_{enc}\rangle$ respectively).
- We achieve this by having having a *code* $\mathcal{C}$, where $\mathcal{C}$ is the set $\mathcal{C} = \{|0_{enc}\rangle, |1_{enc}\rangle\}$ and the errors associated with the encoded qubits project into disjoint sets.



P. Kaye, R. La Flamme, M. Mosca, An Introduction to Quantum Computing (2007)

Correctable Code Ctd...

- We say that a code is correctable if:

Theorem (Knill-Laflamme)

A subspace $\mathcal{C} \subset \mathcal{H}$ is correctable for $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^*$ iff $\exists \Lambda = (\lambda_{ij})$ (a scalar matrix) such that $\forall i, j$:

$$P_{\mathcal{C}} E_i^* E_j P_{\mathcal{C}} = \lambda_{ij} P_{\mathcal{C}}.$$

- Notice the similarity between the definition of correctable and the definition of the rank-k numerical range.
- In terms of operators, we can restate this definition to imply that a code $\mathcal{C}$ is correctable for state ρ which has associated error operator $\mathcal{E}$ iff $\exists$ a recovery operation $\mathcal{R}$ such that $\mathcal{R}(\mathcal{E}(\rho)) = \rho$.

An Application in Quantum Data Error Correction

- A quantum data error correction technique which uses the higher rank numerical range involves using *bi-unitary channels (BUC's)*.

Definition ([1])

A *bi-unitary channel* is a randomized unitary channel $\mathcal{E} = \{V, W\}$ on a Hilbert space $\mathcal{H}$ with an operator-sum representations consisting of two unitaries; so

$$\mathcal{E}(\sigma) = pV\sigma V^* + (1-p)W\sigma W^*, \quad \forall \sigma \in \mathcal{L}(\mathcal{H})$$

for a fixed p with $0 \leq p \leq 1$.

- The p mentioned here is associated with the probability of an error occurring, and it is fixed.

An Application in Quantum Data Error Correction Ctd...

- Choi, Kribs and Życkowski presented the following theorem regarding the correctable code:

Theorem ([4])

Let $\mathcal{C} = V, W$ be a BUC on a Hilbert space $\mathcal{H}$ with dim$(\mathcal{H}) \geq 4$. Then there are 2-dimensional code subspaces $\mathcal{C}$ of $\mathcal{H}$ such that $\mathcal{C}$ is correctable for $\mathcal{E}$.

- After identifying the correctable qubit codes for such channels, it is possible to solve the error correction problem for BUC's on a four-dimensional Hilbert space $\mathbb{C}^4$.

An Application in Quantum Data Error Correction Ctd...

- It turns out that after several steps, the problem reduces down to a single normalized equation of the form $PUP = \lambda P$ for λ and P where U is a single unitary on $\mathcal{H}$.

- The important aspect to realize is that after reducing to the form mentioned above, the eigenvalues of the unitary matrix U will be on the unit circle, which brings us back to the geometry of the higher-rank numerical range.

Bibliography

References

[1] M.D. Choi, D.Kribs, K.Życkowski (2006),
Higher-Rank Numerical Ranges and Compression Problems,
Linear Algebra and it's applications, Issue 418 (2006), pp. 828-839.

[2] M.D. Choi, M. Giesinger, J.A. Holbrook, D.W. Kribs (2006)
Geometry of Higher-Rank Numerical Ranges,
Linear and Multilinear Algebra Vol. 56, Nos. 1-2, Jan—Mar 2008, pp. 53-64.

[3] M.D. Choi, J.A. Holbrook, D.W. Kribs, K.Życkowski (2006)
Higher Rank Numerical Ranges of Unitary and Normal Matrices,
Operators and Matrices, Volume 1, Number 3 (2007), pp. 409Ű426

[4] M.D. Choi, D.Kribs, K.Życkowski (2006),
Quantum Error Correcting Codes From The Compression Formalism,
Reports on Mathematical Physics, Vol 58, Issue 1, August 2006, pp. 77-91.

[5] H.L. Gau, C.K. Li, Y.T. Poon, N.S. Sze (2009)
Quantum Error Correction and Higher rank numerical ranges of normal matrices
arXiv:0902.4869v1

Bibliography

References Ctd...

[6] M.A. Giesinger (2006)
Computational and Convexity Problems for Higher-Rank Numerical Ranges with Applications to Quantum Error Correction,
M.Sc. Thesis - University of Guelph, Dec 2006

[7] K.E. Gustafson, D.K.M. Rao (1991),
Numerical Range - The Field of Values of Linear Operators and Matrices,
Springer-Verlag New York Inc.

[8] P. Kaye, R. La Flamme, M. Mosca (2007)
An Introduction to Quantum Computing,
Oxford University Press, 2007

[9] C.K. Li, N.S. Sze (2008)
Canonical forms, higher Rank numerical ranges, totally isotropic subspaces and matrix equations,
Proceedings of the American Mathematical Society, Volume 136, Number 9 (2008), pp. 3013-3023.

Bibliography

References Ctd...

[10] E. Roldan-Pensado (2008)
Infinite Numerical Range is Convex,
Linear and Multilinear Algebra, 2008, pp. 1 - 3

[11] J.P. Woerdeman (2007)
The Higher Rank Numerical Range is Convex,
Linear and Multilinear Algebra, 2007, Issue 56:1, pp. 65-67.

Acknowledgements

- Professor D.W. Kribs
  Assistant Professor
  Department of Mathematics and Statistics
  University of Guelph

- Professor J. Holbrook
  Professor (Emeritus)
  Department of Mathematics and Statistics
  University of Guelph

- Professor R. Pereira
  Assistant Professor
  Department of Mathematics and Statistics
  University of Guelph

Acknowledgements Ctd...

- N. Johnston
  Ph.D. Candidate
  Department of Mathematics and Statistcs
  University of Guelph

- A. Pasieka
  Ph.D. Candidate
  Department of Physics
  University of Guelph

Acknowledgements Ctd...

- The Fields Institute, Toronto

FIELDS

- Department of Mathematics and Statistics, University of Guelph

UNIVERSITY OF GUELPH

RERUM COGNOSCERE CAUSAS

Department of Mathematics & Statistics

Acknowledgements Ctd...

- NSERC Canada

End

Thank You

Questions or Comments?