Quantum walks with anyons

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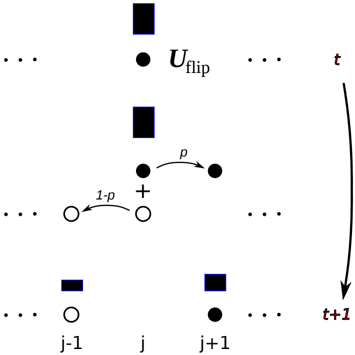
(Based on an idea by G. Brennen and J. Pachos)

Quantum Walks

- Coherent movement of a particle in a discrete lattice
- The direction of movement is decided by a virtual coin
- One unitary time step consists of a coin flipping operation and a subsequent conditional shift operation (see Figure on the next page)

Quantum Walks



The dynamics of one step of the quantum walk. The walker moves to the right with probability p and to the left with probability $1 - p$.

Quantum Walks: Properties

- No correlations in Markovian systems
- Unitary time evolution does not forget the history of the particle $\Rightarrow$ correlations
- The average position of the one-dimensional walker depends linearly on time: $\langle x \rangle \propto t$ (Markovian walks: $\langle x \rangle \propto t^{1/2}$)
- Quantum walks might improve the efficiency of some computational algorithms[1]

[1] J. Kempe: Quantum random walks: an introductory overview, Contemporary Physics **44** (4): 307 (2003)

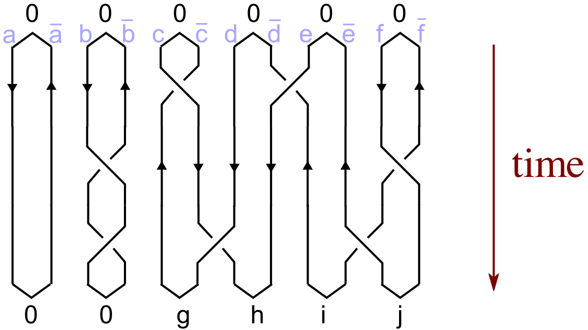
Anyons: What Are They?

- Point like particles in two dimensions
- Could be observed in systems which obey fractional quantum Hall statistics, for example
- The effect of exchange of particles on the wave function is either multiplication by a phase (Abelian anyons) or a transformation of the degenerate ground state (non-Abelian anyons)

Anyons: Time Evolution

- The time evolution of anyons is completely determined by the topological paths the anyons take (*topological order*)
- Paths can be described as a sequence of two-particle exchanges, see Figure on the next page.
- The time evolution can be solved using Skein relations of knot theory

Anyons: Time Evolution



An example of the time evolution for a system of 12 anyons. Particles are created from vacuum with total topological charge 0, exchanged with each other, and adjacent particles are joined together in the end.

Initial State

- One-dimensional walk on an essentially two-dimensional system
- Introduce an additional *fusion space*: $\mathcal{H} = \mathcal{H}_{\text{fusion}} \otimes \mathcal{H}_{\text{space}} \otimes \mathcal{H}_{\text{coin}}$
- Fusion states arise because of the topological degree of freedom which transforms as a result of particle exchange
- The fusion space of the anyons can be regarded as an environment for the spin-1/2 quantum walker

Time Evolution

- The time evolution of anyon walks is generated by the unitary one-step operation
- One step consists of the coin flip and the conditional spatial shift *and braiding* operation:

$$\begin{aligned} |\psi(t)\rangle &= \sum_j \big(B_{j-1,\,j} \otimes |j-1\rangle\langle j| \otimes |{\downarrow}\rangle\langle{\downarrow}| \\ &\quad + B_{j,\,j+1} \otimes |j+1\rangle\langle j| \otimes |{\uparrow}\rangle\langle{\uparrow}|\big) \\ &\quad \big(\mathcal{I}_{\text{fusion}} \otimes \mathcal{I}_{\text{space}} \otimes U_{\text{flip}}\big)|\psi(t-1)\rangle \end{aligned}$$

- The braid matrices $B_{j,\,j+1}$ are determined by the anyon model in question

Time Evolution: Anyon Models

- Anyon models describe the particle types and the fusion rules
- The Hilbert spaces of topological systems grow exponentially as a function of the particle number
- Different anyon models: $SU(2)_k$, Fibonacci, Ising

Interesting Questions

- How do anyon walks compare to Markovian random walks and spin-$1/2$ quantum walks?
- What differences are there between different anyon models?
- Can the decoherence effects be characterized constructively?

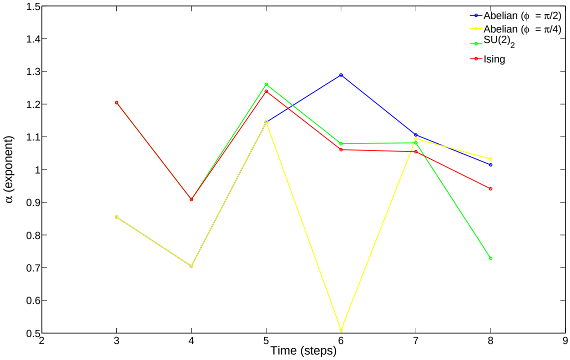
Dispersion

- Dispersion is the standard deviation of the spatial distribution of the walker, or the average position: $\langle x \rangle = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$
- The power $\alpha : \langle x \rangle \propto t^{\alpha}$ characterizes the speed of the walker (random walks: $\alpha = 1/2$, quantum walks $\alpha = 1$)

Dispersion

- In principle, the eigenvalues and -vectors of the one-step operation can be solved and the moments $\langle x^m \rangle$ calculated
- No analytical results for the asymptotic case $t \to \infty$ have been obtained so far
- Simulational results can be obtained for the case $t \approx 10$ (see Figure on next page)

Dispersion: Simulation Results



The power α that characterizes the speed of the walker. Asymptotically, $\alpha = 0.5$ for Markovian walks and $\alpha = 1$ for spin-$1/2$ walks.

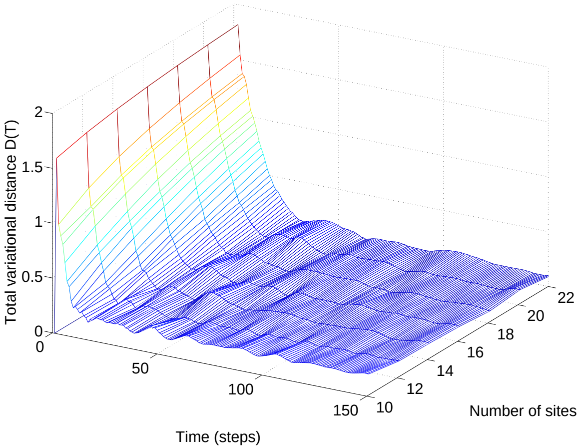
Dispersion

- Simulations show a time dependence of around $t^{0.8}$, which is between Markov walks and quantum walks

Mixing Time: Periodic Walks

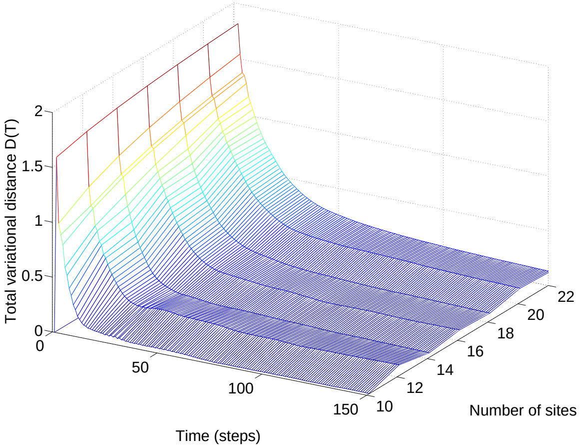
- Mixing time describes how rapidly a *periodic* walk settles to a uniform (totally random) distribution
- Defined as the time that it takes for the *total variational distance* to converge within some constant interval
- Total variational distance: $D(T) = \sum_x \left| \overline{P(x,T)} - P_\mu \right|$, where P_μ is the uniform distribution $(1/N)$ and $\overline{P(x,T)}$ is the *time-averaged* probability at each site on a given time step
- The periodic boundary condition for the fusion states can be expressed as a product of the braid matrices for all the pairs

Mixing Time: Simulation Results



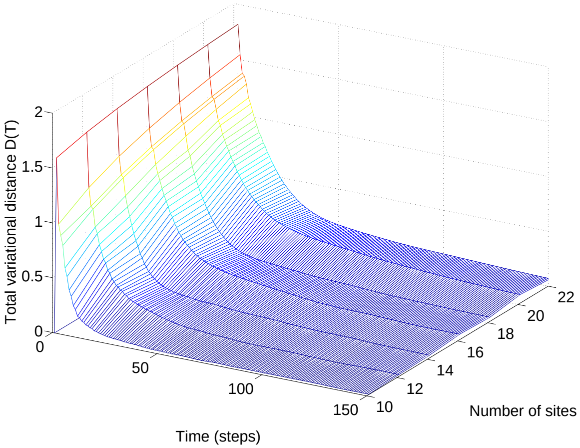
The total variational distance for the periodic spin-1/2 walk with 10–22 sites.

Mixing Time: Simulation Results



The total variational distance for the periodic $SU(2)_2$ walk.

Mixing Time: Simulation Results



The total variational distance for the periodic Ising walk.

Decoherence Effects

- The exchange operation acts as a non-local source of decoherence
- The fusion space $\mathcal{H}_{\text{fusion}}$ can be seen as an environment for the spin-1/2 walker, degrading the correlations between the spatial and coin states
- Markovian walks can be seen as the lower bound and the spin-1/2 walk as the upper bound of performance
- Decoherence model: assign spins to each site and do an operation on them when they are exchanged

Conclusions

- Anyons are particles in two dimensions and their wavefunction experiences a nontrivial transformation upon permutations
- Algorithmic properties of anyon walks are between Markovian random walks and spin-1/2 quantum walks
- Anyon walks can be simulated with only up to 20–30 anyons

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http://web.science.mq.edu.au/~llehman/Anyon_walks.pdf