The Geometry of Anticoherent Spin States

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Outline

1. Anticoherence
   - Definition
   - The Majorana Representation
2. Spherical Designs
   - Symmetric Designs
   - Connection to Anticoherence
3. Application
   - Symmetric State Entanglement

Coherent States

- The most "classical" spin sates.
- Can be identified with a classical spin direction: $|\psi_{\mathbf{n}}\rangle \sim \mathbf{n}$.
- For a spin-s system in the coherent state $|\psi_{\mathbf{n}}\rangle$,

$$\langle \mathbf{S} \rangle = (\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle) = s\hbar\mathbf{n}.$$

Anticoherent States

- Cannot be associated with a classical spin direction.
- Point "nowhere" in the mean: $\langle \mathbf{S} \rangle = \langle \psi | \mathbf{S} | \psi \rangle = \mathbf{0} \Leftrightarrow \langle \mathbf{n} \cdot \mathbf{S} \rangle$ being independent of $\mathbf{n}$.

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Definition

A state $|\psi\rangle$ is anticoherent if $\langle \mathbf{n} \cdot \mathbf{S} \rangle$ is independent of $\mathbf{n}$.

Example: $|\psi\rangle = |s = 1, m_z = 0\rangle$ satisfies $\langle \mathbf{S} \rangle = \mathbf{0}$.

Higher Orders of Anticoherence

- Two labs L_1 and L_2:
  - L_1 measures $\mathbf{n}_1 \cdot \mathbf{S}$ many times on a system prepared in the anticoherent state $|s = 1, m_z = 0\rangle$.
  - L_2 measures $\mathbf{n}_2 \cdot \mathbf{S}$ many times on the same type of system prepared in the same state.
- Directional dependence of $\langle(\mathbf{n} \cdot \mathbf{S})^2\rangle$.

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Definition

A state $|\psi\rangle$ is anticoherent to order t if $\langle(\mathbf{n} \cdot \mathbf{S})^k\rangle$ is independent of $\mathbf{n}$ for $k = 1, ..., t$.

THE MAJORANA REPRESENTATION

Spin-1/2

$$|\psi\rangle = a_\uparrow|\uparrow\rangle + a_\downarrow|\downarrow\rangle$$

ratio $\quad r = \frac{a_\downarrow}{a_\uparrow}$

root of $\quad M(z) = a_\uparrow z - a_\downarrow$

Stereographic projection:

$$\mathbf{V}(r) = \left(\frac{2\Re(r)}{1+|r|^2}, \frac{2\Im(r)}{1+|r|^2}, \frac{1-|r|^2}{1+|r|^2}\right)$$

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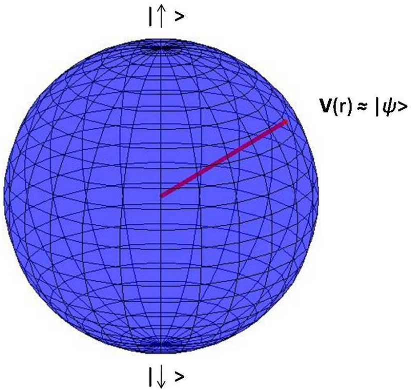

Spin-1

$$|\psi\rangle = a_1|1\rangle + a_0|0\rangle + a_{-1}|-1\rangle$$

$$\Updownarrow$$

$$M(z) = a_1 z^2 - \sqrt{2}a_0 z + a_{-1}$$

two roots: r_1, r_2

$$\Updownarrow$$

two points on $\mathcal{S}^2$: $\mathbf{V}(r_1), \mathbf{V}(r_2)$

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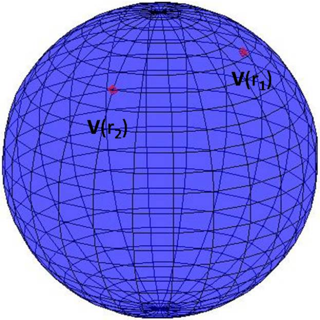


States $\Leftrightarrow$ points on $\mathcal{S}^2$

$$Spin - s: \qquad |\psi\rangle = \sum_{m=-s}^{s} a_m |m\rangle \ \in \mathcal{H}_{2s+1}$$

$$\Updownarrow$$

$$M(z) = \sum_{m=-s}^{s} a_m (-1)^{m-s} \sqrt{\binom{2s}{s+m}} z^{s+m} \qquad degree = 2s$$

$$\Updownarrow$$

$$\{r_i\}_{i=1}^{2s}$$

$$\Updownarrow$$

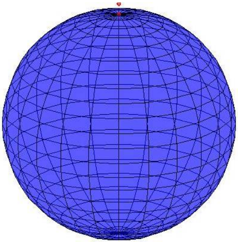
2s points on $\mathcal{S}^2$ via stereographic projection

Spin-1: special examples

Basis sates: $|1\rangle, |0\rangle, |-1\rangle$

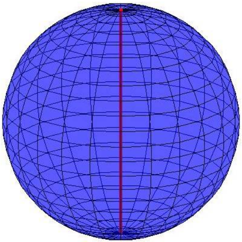
$|1\rangle$

spin up - coherent



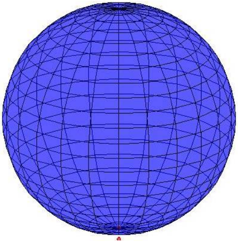
$|0\rangle$

anticoherent



$|-1\rangle$

spin down - coherent



Majorana Picture

- Coherent states: one point with multplicity $2s$.
- Anticoherent states: points are "nicely" spread over the sphere.

SPHERICAL DESIGNS

Equivalent Definitions

Definition

Let $t \in \mathbb{N}$. A finite (non-empty) subset $X \subset \mathcal{S}^2$ is a spherical t-design if

$$\frac{1}{|X|} \sum_{\mathbf{x} \in X} f(\mathbf{x}) = \frac{1}{\mu(\mathcal{S}^2)} \int_{\mathcal{S}^2} f(\mathbf{x}) d\mu(\mathbf{x})$$

holds for any polynomial $f(\mathbf{x})$ of deg $\leq t$.

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holds for any polynomial $f(\mathbf{x})$ of deg $\leq t$.

Observation

$X \subset \mathcal{S}^2$ is a spherical t-design if the kth moments of X are invariant under $\mathcal{O}(3)$ for $k = 1, ..., t$.

A spherical 1-design is a set whose vector sum is zero.

Spherical Designs vs. Anticoherent States

Definition

A state $|\psi\rangle$ is anticoherent to order t if $\langle(\mathbf{n}\cdot\mathbf{S})^k\rangle$ is independent of $\mathbf{n}$ for $k = 1, ..., t$.

$\Updownarrow$ via Majorana Representation ?

Observation

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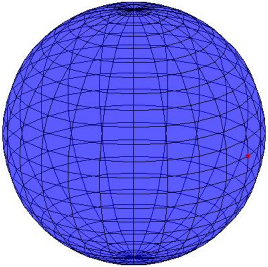
Orbits

Can generate spherical designs using orbits of finite subgroups of $\mathcal{O}(3)$. Take $G = \{I, R_{2\pi/3}, R_{4\pi/3}\}$ and $\mathbf{x} = (1, 0, 0)$.

$\{I\mathbf{x}\}$



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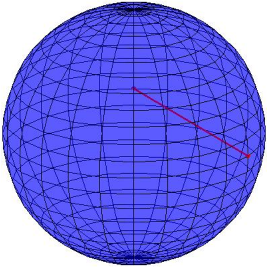

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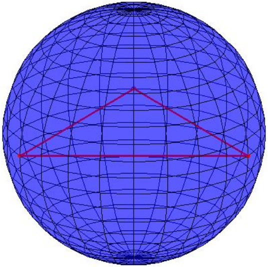

Main Result

Theorem

Let $\mathbf{x} \in \mathcal{S}^2$ and let G be a (non-empty) finite subgroup of $\mathcal{O}(3)$. If $G\mathbf{x}$ is a spherical t-design, then the state $|\psi\rangle$ whose Majorana representation is $G\mathbf{x}$ is anticoherent to order t.

Some Curious Geometry

Spin-2

- $G = Sym(T)$
- $G\mathbf{x}$ is a spherical 2-design for all $\mathbf{x} \in \mathcal{S}^2$.
- Gives anticoherent states of order 2.
- Example:

$$|\psi\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|1\rangle - |-2\rangle)$$

Some Curious Geometry

Spin-4

- $G = Sym(C)$
- $G\mathbf{x}$ is a spherical 3-design for all $\mathbf{x} \in \mathcal{S}^2$.
- Gives anticoherent states of order 3.
- Example:

$$\frac{1}{2\sqrt{6}}(\sqrt{5}|4\rangle + \sqrt{14}|0\rangle + \sqrt{5}|-4\rangle)$$

Some Curious Geometry

Spin-10

- $G = Sym(D)$
- $G\mathbf{x}$ is a spherical 5-design for all $\mathbf{x} \in \mathcal{S}^2$.
- Gives anticoherent states of order 5.
- Example:

$$a|10\rangle + b|5\rangle + |0\rangle - b|-5\rangle + a|-10\rangle)$$

SYMMETRIC STATE ENTANGLEMENT

Symmetric States

Symmetric states on n qubits:

$$|\psi_{sym}\rangle = N \sum_{\sigma \in \mathcal{S}_n} |r_{\sigma(1)}\rangle |r_{\sigma(2)}\rangle \cdots |r_{\sigma(n)}\rangle$$

Example:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = N \sum_{\sigma \in \mathcal{S}_3} |r_{\sigma(1)}\rangle |r_{\sigma(2)}\rangle |r_{\sigma(3)}\rangle$$

where

$$\begin{aligned} |r_1\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |r_2\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle + e^{i2\pi/3}|1\rangle) \\ |r_3\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle + e^{i4\pi/3}|1\rangle) \end{aligned}$$

Geometric Measure of Entanglement

$$E_G(|\psi\rangle) = 1 - \max_{|\phi\rangle \in Prod} |\langle\psi|\phi\rangle|^2$$

- $E_G \geq 0$ with equality iff $|\psi\rangle$ is a product state.
- Maximum product state $|\phi\rangle$ can be found in symmetric subspace.
- Maximum product state $|\phi\rangle$ is *necessarily* symmetric. [Hübener, Kleinmann, et. al, 2009]

Maximum Entanglement

- For symmetric states $|\psi\rangle$ and symmetric product states $|\phi\rangle = |a\rangle|a\rangle\cdots|a\rangle$,

$$|\langle\psi|\phi\rangle|^2 \propto \prod_{i=1}^{n} |\langle r_i|a\rangle|^2 = \prod_{i=1}^{n} \frac{||\mathbf{V}(r_i) - \mathbf{V}(a^*)||^2}{4}$$

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- For fixed n, symmetric states $|\psi\rangle$ which maximize $E_G(|\psi\rangle)$ satisfy

$$\min_{|\psi\rangle\in Sym(\mathcal{H})} \max_{\mathbf{x}\in\mathcal{S}^2} \prod_{i=1}^{n} ||\mathbf{V}(r_i) - \mathbf{x}||^2$$

.

3-qubits

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

- "Maximally entangled" via E_G. [Tamaryan, Wei, Park, 2009]

$$|r_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
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$$\text{roots: } 1, e^{i2\pi/3}, e^{i4\pi/3}$$

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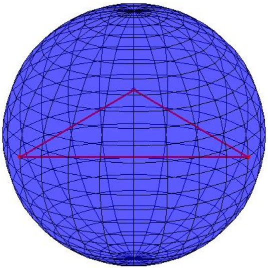
-of unity!

$|GHZ\rangle$



Spherical 1-design

Final Conjecture

- Symmetric state with highest geometric entanglement is that whose n points are mutually maximally distant ($n > 3$). [Markham, 2007]

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Conjecture

Fix $n \in \mathbb{N}$. Let τ_n be the largest value of t for which an n-point spherical t-design exists. Symmetric states that are spherical τ_n-designs maximize E_G.

For Further Reading I

J. Zimba.
"Anticoherent" Spin States via the Majorana Representation.
Electronic Journal of Theoretical Physics, No.10, 143-156, 2006.

R. Hübener, M. Kleinmann, T.C. Wei, C. González-Guillén and O. Gühne.
The geometric measure of entanglement for symmetric states
arXiv:0905.4822v2, 2009.

S. Tamaryan, T.C. Wei, D. Park.
Maximally entangled three-qubit states via geometric measure of entanglement
arXiv:0905.3791v2, 2009.

For Further Reading II

D. Markham
Symmetric State Entanglement in the Majorana Representation
http://www.pps.jussieu.fr/ djhm

Thank you

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  NSERC CRSNG
- Fields Institute

  FIELDS
- Thank you all!