

Mapping cones of positive maps.

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K , H finite dim. Hilbert spaces.

$A \subseteq B(K)$ operator system, if.
self-adjoint linear subspace.

$B(A, H) = (\text{Crep } B(A, H))^+$ =

linear space of linear Crep.

positive linear maps $\phi: A \rightarrow B(H)$

$$\tilde{\phi}(a \otimes b) = \text{Tr}(\phi(a)b^*)$$

$a \otimes b \in A \otimes B(H)$.

The map $\phi \mapsto \tilde{\phi}$ is an isom.
between $B(A, H)$ and $(A \otimes B(H))^*$.

$\phi \geq 0$ iff $\tilde{\phi}$ is positive on the cone

Let $P(H) = B(B(H), H)^+$.

A cone $\mathcal{C} \subset P(H)$ is a

mapping cone if $\phi \in \mathcal{C}$

$\Rightarrow \alpha \circ \phi \circ \beta \in \mathcal{C}$ whenever
 $\alpha, \beta \in P(H)$ are completely
positive, i.e. $\alpha, \beta \in CP(H)$.

Example.

$P(H), CP(H)$.

$S(H) = \{\phi : \phi(x) = \sum_i c_i \phi_i(x) b_i ;$
 c_i : state on $B(H)$, $b_i \in B(H)^+$).

$\phi \in S(H)$ is called superpositive,
or entanglement breaking.

Lemma \mathcal{C} mapping cone \Rightarrow

Let $\mathbb{1}$ denote the identity map and

$$P(A, G) = \{x \in A \otimes B(H) : x \text{ s.a.}$$

$$\operatorname{re} x > 0 \quad \forall x \in G\},$$

is a proper convex cone in $A \otimes B(H)$ containing $A^+ \otimes B(H)^+$.

$\phi \in B(A, H)$ is G -positive

if $\tilde{\phi}$ is positive on $P(A, G)$.

Thm: $A \subset B \subset B(K)$ operator systems. $\phi \in B(A, H)^+$ G -positive
 $\Rightarrow \phi$ has a G -positive extension in $B(B, H)^+$.

4. Advanced Artinian H .

Ex: C^* map.

If $\phi \in P(H)$ ϕ^* and ϕ^t are defined by:

$$\text{Tr}(\phi(a)b) = \text{Tr}(a\phi^*(b))$$

$$\phi^t(a) = t \circ \phi \circ t(a) = \phi(a^t)^t.$$

We say \mathcal{C} is symmetric

$$\text{if } \mathcal{C} = \mathcal{C}^t = \mathcal{C}^*, \text{ where}$$

$$\mathcal{C}^t = \{\phi^t : \phi \in \mathcal{C}\},$$

$$\mathcal{C}^* = \{\phi^* : \phi \in \mathcal{C}\}.$$

Then Assume \mathcal{C} is symmetric.

Let $P(\mathcal{C})$ denote the closed cone in $B(A, H)$ generated by all maps of the form $\alpha \circ \psi$, $\alpha \in \mathcal{C}$, $\psi \in CPCA(H)$. Then $\alpha \in B(A, H)$ is \mathcal{C} -positive $\Leftrightarrow \alpha \in P(\mathcal{C})$

Cor. Assume $K \geq H$ and \mathcal{C} symmetric. $\phi \in B(A, H)^+$. Then ϕ is \mathcal{C} -positive iff $\phi = \psi|_A$ with $\psi \in \mathcal{C}$.

Proof: By the extension theorem assume $A = B(H)$.

Then $P(\mathcal{C}) = \mathcal{C}$, so by Theorem $\phi \in \mathcal{C}$ iff ϕ is \mathcal{C} -pos.

Let $\phi \in B(B(K), H)$. Let (e_{ij}) be a complete set of matrix units for $B(K)$. Then the Choi matrix C_ϕ is

$$C_\phi = \sum e_{ii} \otimes \phi(e_{ii}) \in B(K) \otimes B(H)$$

Lemma (i) $C_{\phi^{\otimes k}} = C_{\tilde{\phi}}^t$ is the density matrix for $\tilde{\phi}$.

(ii) ϕ is C.P iff $\tilde{\phi}$ is positive.

Ad(i). By Choi, ϕ is CP iff

$$C_{\phi} \geq 0. \text{ i.e. } C_{\phi}^* = C_{\phi^{\otimes 2}} \geq 0 \text{ i.e. } \tilde{\phi} \geq 0.$$

Duality

$S \subset B(B(K), H)^+$. The dual cone S^o of S is

$$S^o = \{ \phi \in B(B(K), H)^+ : \operatorname{Tr}(C_\phi C_\psi) \geq 0 \text{ for all } \psi \in S \}.$$

Thus let $P_G(K)$ denote the cone of G -positive maps in $B(B(K), H)^+$.

Let $\phi \in B(B(K), H)$. Then F.A.E.

$$(i) \quad \phi \in P_G(K)^0$$

$$(ii) \quad 2 \otimes \alpha(C_\phi) \geq 0 \quad \forall \alpha \in G^*$$

$$(iii) \quad \tilde{\phi} \circ (2 \otimes \alpha^*) \geq 0 \quad \forall \alpha \in G$$

$$(iv) \quad \alpha \circ \phi \in CP(B(K), H) \quad \forall \alpha \in G^*$$

$P_G(H)^0$ is a mapping
cone. (note: here $K=H$).

PL. Let α, ϕ as in (iv). Let
 $\rho \in CP(H)$. Then

$$\alpha \cdot (\rho \circ \phi) = (\alpha \cdot \rho) \circ \phi \in CP(B(K), H)$$

because $\alpha \circ \rho \in G^*$.

$$\alpha \circ (\phi \circ \rho) = (\alpha \circ \phi) \circ \rho \quad \text{is CP.}$$

Then Assume $K=H$ and G symmetric.
 Let ρ be a linear functional on $B(H \otimes H)$ with density operator \hat{h} . Then
 F.A.E.

- (i) $\rho = \tilde{\phi}$ with $\phi \in G^0$.
- (ii) $\rho(C_C\alpha) \geq 0 \quad \forall \alpha \in G$.
- (iii) $z \otimes \alpha(\hat{h}) \geq 0 \quad \forall \alpha \in G$.
- (iv) $\rho \circ (z \otimes \alpha) \geq 0 \quad \forall \alpha \in G$
- (v) ρ is positive on the cone

$$\{x \in B(H \otimes H) : z \otimes \alpha(x) \geq 0 \forall \alpha \in (G^0)^*\},$$

$$(\text{or } \langle \alpha \tilde{\phi}^*(x), \beta \rangle \geq 0 \forall \alpha, \beta \in G^0)$$

Separable States on $B(H \otimes H)$

By Horodecki, Horodecki, and Ruskai

$\phi \in \text{SCH}$) iff $\tilde{\phi}$ is separable state.

(pf. $\phi(a) = \sum \omega_i(a) b_i$; then

$$\begin{aligned}\tilde{\phi}(a \otimes b) &= \text{Tr}(\phi(a)b^*) = \\ &= \sum \text{Tr}(\omega_i(a)b_i b^*) \\ &= \sum \omega_i(a) \text{Tr}(b_i b^*) \\ &= \sum \omega_i(a) p_i(b). \\ &= \sum \omega_i \otimes p_i(a \otimes b).\end{aligned}$$

Converse is proved by going backwards.)

$\therefore \rho$ is separable iff $\rho = \tilde{\phi}$ with $\phi \in \text{SH}$.

By minimality of SCH) and
max. of PCH). $\text{SCH}^\circ = \text{PCH}$)

and $S(H) = P(H)^o$.

$\therefore \rho$ is separable iff $\rho = \phi$, $\phi \in P(H)$
 so by Thm. iff $\text{rank}(\rho) \geq 0$
 $\forall \alpha \in P(H)$.

(This is the Horodecki Thm.)

By (v) in thm. ρ is separable
 iff ρ is positive on the cone
 $\{x \in B(H \otimes H) : \text{rank}(x) \geq 0 \quad \forall \alpha \in S(H)\}$
 $= \{x \in B(H \otimes H) : \text{rank}(x) \geq 0 \quad \forall \text{ states } \omega \text{ on } B(H)\}$

(This is also well-known. $\Leftrightarrow \phi \in B(L^2(H))^\dagger$.)

Decomposable maps and PPT-state

A map $\phi \in B(A, H)^+$ is decomposable if $\phi = \phi_1 + \phi_2$ with ϕ_1 c.p. and ϕ_2 copositive, i.e. $\phi_2 = t \circ \psi$ with ψ c.p.

A state ρ on $A \otimes B(H)$ is a PPT-state if $\rho \circ (\tau \otimes t) \geq 0$.

Prop ϕ is both c.p. and copos. iff $\tilde{\phi}$ is a PPT-state (when normalized).

$$\begin{aligned}
 \text{Ex. } \tilde{\phi}(\tau \otimes t(a \otimes b)) &= \\
 &= \tilde{\phi}(a \otimes b^*) + \text{Tr}(\phi(a)b) = \\
 &= \text{Tr}(t \circ \phi(a)b^*) = t \circ \tilde{\phi}(a \otimes b). \\
 \therefore \tilde{\phi} \text{ is PPT iff } \tilde{\phi} \text{ and } \tilde{\phi} \text{ are pos. iff}
 \end{aligned}$$

Let \mathcal{C} denote the mapping cone

$$\mathcal{C} = \text{CP(H)} \cap \text{cpos}(H).$$

ϕ is c.p iff $C_\phi \geq 0$ iff

$$\text{Tr}(C_\phi C_{\phi^*}) \geq 0 \text{ & } C_{\phi^*} \geq 0$$

$$\phi \in \text{CP(H)}^\circ,$$

$$\therefore \text{CP(H)} \supset \text{CP(H)}^\circ,$$

$$(\text{cpos}(H))^\circ = \text{cpos}(H).$$

Therefore

$$\mathcal{C}^\circ = \text{CP(H)} \vee \text{cpos}(H).$$

$\boxed{\therefore \phi \text{ is decomposable iff } \phi \in \mathcal{C}^\circ}$
and $\rho = \tilde{\psi}$ is PPT iff $\psi \in \mathcal{C}$

The time on states becomes:

Thm. 6 as above. ρ state on $B(H \otimes H)$ with density operator h .
Then F.A.E.

- (i) $\rho = \phi$ with ϕ decomposable.
- (ii) $\rho(C_\alpha) \geq 0 \quad \forall \alpha \in \mathcal{C}$
- (iii) $2\phi\alpha(h) \geq 0 \quad \forall \alpha \in \mathcal{C}$
- (iv) $\rho \circ (2\phi\alpha) \geq 0 \quad \forall \alpha \in \mathcal{C}$
- (v) ρ is positive on the cone
 $\{x \in B(H \otimes H) : 2\phi\alpha(x) \geq 0 \text{ and decomposable}\}$
- (vi) ρ is positive on the cone
 $\{x \in B(H \otimes H) : 2\phi\alpha(x) \geq 0\}$

also have

- (vii) $2\phi\alpha(h) \geq 0 \iff \text{rank}(h) \geq 0$.

k -positive and k -superpositive maps

(joint work with L. Skowronek and
K. Życzkowski)

Let $\phi \in P(H)$, $\tau_k = \text{identity map}$
on $B(C^k)$.

ϕ is k -positive if $\phi \otimes \tau_k \geq 0$

If $\phi \in CP$ and $\phi \otimes \sum v_i^* x v_i$ and
rank $v_i \leq k - r_i$, then ϕ is
 k -superpositive, i.e. $\phi \in SP_k(H)$

(so $k \geq \dim H \Rightarrow \phi \in CP$, and $SP_1(H) = SH$)

Let $P_k(H)$ = set of k -positive op

Let $P_k^t(H) = \{ t \circ \phi : \phi \in P_k(H) \}$

$SP_k^t(H) = \{ t \circ \phi : \phi \in SP_k(H) \}$.

Let $\phi \in P(H)$. Then ϕ is

(k,m) -decomposable if $\phi \in D_{km}(H) = P_k(H) \vee P_m^t(H)$.

(k,m) -positive if $\phi \in P_{km}(H) = P_k(H) \cap P_m^t(H)$

(k,m) -superpositive if $\phi \in SP_{km}(H) = SP_k(H) \cap SP_m^t(H)$

atomic if $\phi \notin D_{aa}(H)$.

Example of atomic maps are different generalizations of the Choi map in $\text{PL}(C^*)$ and projective maps onto spin factors. (Ha, Tomiyama, Kye, St. Robertson)

There is a generalization of Choi's characterization of CP maps:

$\phi \in \text{P}_e(H)$ iff C_ϕ is k -block positive
i.e. $(\sum_i z_i \otimes q_i, C_\phi \sum_i z_i \otimes q_i) \geq 0$
whenever $z_i, q_i \in H$, $i=1, \dots, k$.

Prop: $P_k(H)^0 = SP_k(H)$

$D_{km}(H)^0 = SP_{km}(H)$

Then $\phi \in P(H)$. F.A.E.

(i) $\phi \in P_k(H)$

(ii) $\psi \circ \phi \in SP_k(H) \vee \psi \in SP_k(H)$

(iii) $\psi \circ \phi \in CP(H) \vee \psi \in SP_k(H)$

(iv) $\Im(\psi(C_\phi)) \geq 0 \vee \psi \in SP_k(H)$

(v) $\Im \circ (\Im \psi) \geq 0 \vee \psi \in SP_k(H)$

There is a similar result for $\phi \in SP_k(H)$.