LOCC in Operator Algebra Language

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Basic set-up

$$\mathcal{H} = \mathbf{C}_d \qquad \mathfrak{A} = \mathcal{B}(\mathcal{H}) = M_d$$

will use tensor products $\mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \ldots \mathfrak{A}_n$ or $\mathfrak{A} \otimes \mathfrak{B}$

local means acts only on components of a tensor product

$\mathfrak{C} \subset M_d$ denotes classical algebra of diagonal matrices in some M_d

algebra associated with "Alice" $\mathfrak{A}_A$ or $\mathfrak{A}$

algebra associated with "Bob" $\mathfrak{A}_B$ or $\mathfrak{B}$

identify pure state with vector $|\psi\rangle \in \mathcal{H}$ or better

     rank one projection $\rho = |\psi\rangle\langle\psi|$

Mixed states and Entanglement

Mixed state is convex comb of pure $\rho = \sum_k p_k |\chi_k\rangle\langle\chi_k|$

$|\chi_k\rangle$ need not be O.N. – not nec spectral decomp.

Identify state with density matrix ρ, i.e., $\rho > 0$, $\quad \text{Tr}\,\rho = 1$

defines pos lin fctnl $\mathfrak{A} \mapsto \mathbf{C}$ given by $A \mapsto \text{Tr}\,\rho A$

pure $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ entangled if it is not product $|\phi_A \otimes \phi_B\rangle$

Def: ρ is separable if convex comb of prods $\quad \rho = \sum_k p_k |\phi_k^A \otimes \phi_k^B\rangle$

$|\phi_k\rangle$ need not be O.N. - spectral decomp not prod in general

Pure $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ is maximally entangled if $\text{Tr}_B |\psi\rangle\langle\psi| = \frac{1}{d} I_A$

Examples: $|\psi\rangle = \sum_k \frac{1}{\sqrt{d}} e^{i\theta_k} |\phi_k^A \otimes \chi_k^B\rangle \qquad d_A = d_b = d.$

Can find O.N. basis for $\mathcal{H} \otimes \mathcal{H}$ consisting of max entang states.

Example: Teleportation

$d = 2$ Max entangled Bell state: e-bit or EPR pair

Def: $|\beta_k\rangle = (I \otimes \sigma_k)|\beta_0\rangle$ $\qquad |\beta_0\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$

For prod of qubits $\mathcal{H}_{A'} \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, i.e., all $\mathcal{H} = \mathbf{C}_2$

$$|\phi \otimes \beta_0\rangle = \sum_{k=0}^{3} \tfrac{1}{2}|\beta_k\rangle \otimes \sigma_k|\phi\rangle$$

- Alice and Bob share max entang $|\beta_0\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
- Alice also has unknown state $|\phi\rangle \in \mathcal{H}_{A'}$ get state $|\phi\rangle \otimes |\beta_0\rangle$
- A makes "Bell" meas.; gets one of $|\beta_k\rangle \otimes \sigma_k|\phi\rangle$ each with prob $\frac{1}{4}$
- Alice learns k from meas – calls and tells Bob to apply σ_k
- Bob ends up with $\sigma_k^2|\phi\rangle = |\phi\rangle$ exactly what Alice had in $\mathcal{H}_{A'}$.

LOCC

Teleportation transfers $|\phi\rangle$ from A' to B using only LOCC

e-bit or EPR pair is important resource in quant info proc.

LOCC = Local Operations and Classical Communication

$$\mathfrak{A}_A \otimes \mathfrak{A}_B \qquad \text{or} \qquad \mathfrak{A} \otimes \mathfrak{B}$$

LO just means a trace-decreasing CP map of form $\Phi_A \otimes \Phi_B$

$$\Phi_A : \mathfrak{A} \mapsto \mathfrak{A}' \qquad \Phi_B : \mathfrak{B} \mapsto \mathfrak{B}'$$

not all authors agree – can be more restrictive wlog

but lose flavor of process

to explain CC review measurement

Quantum basics and von Neumann measurement

Fund Postulate of Q.M.: Observable represented by self-adj op A

spectral decomp $A = \sum_k a_k E_k = \sum_k a_k |\alpha_k\rangle\langle\alpha_k|$

Measurement of A with system in some state ψ.

     (i) get some e-value   (only possibility)

     (ii) leave system in e-state α_k

     (iii) probability is $|\langle\alpha_k, \psi\rangle|^2 = \operatorname{Tr} E_k|\psi\rangle\langle\psi|$

Write $|\psi\rangle = \sum_k c_k|\alpha_k\rangle$ as a superposition of e-states, $c_k = \langle\alpha_k, \psi\rangle$

Coefficients c_k in superpos. give probs $|c_k|^2$ not classical

Average result of meas in state $|\psi\rangle$ is $\langle\psi, A\psi\rangle = \operatorname{Tr} A|\psi\rangle\langle\psi|$

Av result of meas in mixed state $\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|$ is $\operatorname{Tr} A\rho$

set $\{E_k\}$ orthog projections $E_j E_k = E_k \delta_{jk}$ with $\sum_k E_k = I$ called

von Neumann measurement or projection valued measure (PVM)

corresponds to "yes-no" experiment (e.g., polarization filter)

CPT map $\Omega_{\mathcal{M}} : \mathfrak{A} \mapsto \mathfrak{C}$ givees result of PVM or vN measurement

$$\Omega_{\mathcal{M}} : \rho \mapsto \sum_j E_j \rho E_j = \sum_j |\alpha_j\rangle\langle\alpha_j, \rho\,\alpha_j\rangle\langle\alpha_j| = \sum_j E_j \operatorname{Tr} \rho\, E_j$$

QC quantum-classical $\{E_j\}$ O.N. $\Rightarrow$ output in subalg iso to $\mathfrak{C}$

Now consider two non-commuting observables

$$A = \sum_j a_j |\alpha_j\rangle\langle\alpha_j| = \sum_j a_j E_j, \qquad B = \sum_k b_k |\beta_k\rangle\langle\beta_k| = \sum_k b_k F_k$$

- measure A, then B ends in e-state $|\beta_k\rangle$ or F_k of B

- measure B, then A ends in e-state $|\alpha_j\rangle$ or E_j of A

Measure B, then A ends with $F_k \mapsto \Omega_{\mathcal{M}}(F_k) = \sum_j E_j F_k E_j$

Quantum measurement: POVM

$$\sum_{jk} E_j F_k E_j = \sum_j E_j I E_j = I$$

$\{E_j F_k E_j\}$ example of POVM positive operator valued measurement

Def: (Davies-Lewis) POVM $\mathcal{M} = \{G_m\} \quad G_m > 0, \ \sum_m G_m = I$

Result of POVM depends on order in which G_m performed

QC map using instrument with class "pointer" $|f_m\rangle$ O.N.

$$\begin{aligned}
\Omega_{\mathcal{M}} : \rho \ &\mapsto \ \sum_m (\mathrm{Tr}\,\rho\, G_m)|\phi_m\rangle\langle\phi_m| \otimes |f_m\rangle\langle f_m| \\
&= \ \bigoplus_m (\mathrm{Tr}\,\rho\, G_m)\,|\phi_m\rangle\langle\phi_m|
\end{aligned}$$

$$\Omega_{\mathcal{M}} : \mathfrak{A} \mapsto \mathfrak{A} \otimes \mathfrak{C} \simeq \bigoplus \mathfrak{A}$$

Classical Communication

Recall POVM meas. $\qquad \Omega_{\mathcal{M}} : \mathfrak{A} \mapsto \mathfrak{A} \otimes \mathfrak{C} \simeq \bigoplus \mathfrak{A}$

have state $\rho_{AB} \in \mathfrak{A} \otimes \mathfrak{B}$

$$(\Omega_{\mathcal{M}} \otimes \mathcal{I})(\rho_{AB}) = \bigoplus_M |\phi_m\rangle\langle\phi_m| \otimes \mathsf{Tr}_A \, \rho_{AB} \, G_m$$

local meas $(\Omega_{\mathcal{M}} \otimes \mathcal{I}) : \mathfrak{A} \otimes \mathfrak{B} \mapsto (\mathfrak{A} \otimes \mathfrak{C}) \otimes \mathfrak{B} \simeq \mathfrak{A} \otimes \mathfrak{C} \otimes \mathfrak{B}$

math trivial equiv $(\mathfrak{A} \otimes \mathfrak{C}) \otimes \mathfrak{B} \simeq \mathfrak{A} \otimes \mathfrak{C} \otimes \mathfrak{B} \simeq \mathfrak{A} \otimes (\mathfrak{C} \otimes \mathfrak{B})$

class algebra is shared – gives (one-way) classical communication

  one-way: A or B does all measurements, e.g., $\Omega_{\mathcal{M}_A} \otimes \mathcal{I}_B$

  two-way: either A or B can measure, $\Omega_{\mathcal{M}_A} \otimes \mathcal{I}_B$ or $\mathcal{I}_A \otimes \Omega_{\mathcal{M}_B}$

next LO can be conditioned on classical algebra

$$|\phi \otimes \beta_0\rangle\langle\phi \otimes \beta_0| \xrightarrow{\Omega_{\mathcal{M}_{A'A}-\text{Bell}}} \frac{1}{2}\bigoplus_k |\beta_k\rangle\langle\beta_k| \otimes \sigma_k|\phi\rangle\langle\phi|\sigma_k$$

$$= \frac{1}{2}\bigoplus_k |\beta_k\rangle\langle\beta_k| \otimes \Gamma_k(|\phi\rangle\langle\phi|)$$

$$\xrightarrow{\mathcal{I}_{A'A}\otimes(\oplus\Gamma_k)} \left(\frac{1}{2} \oplus_k |\beta_k\rangle\langle\beta_k|\right) \otimes |\phi\rangle\langle\phi|$$

$$\Gamma_k(\rho) \equiv \sigma_k \rho \sigma_k^* \quad \text{unitary conj}$$

More gen CC step $(\Omega_{\mathcal{M}} \otimes \mathcal{I})$: yields $\oplus_k \rho_k \in \oplus \mathfrak{A} \otimes \mathfrak{B}$

Apply cond LO of form $\mathcal{I} \otimes \Phi_B = \mathcal{I} \otimes \oplus_k \Phi_k$ with $\Phi_k : \mathfrak{B} \mapsto \mathfrak{B}'$

Distillation

Typical situation:

$$\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \ldots \mathfrak{A}_n = \mathfrak{A}_1^{\otimes n} \qquad \mathfrak{B} = \mathfrak{B}_1^{\otimes n}$$

$\mathfrak{A} \otimes \mathfrak{B}$ iso to $(\mathfrak{A}_1 \otimes \mathfrak{B}_1)^{\otimes n}$.

start with n copies of $\rho \equiv \rho_{AB} \in \mathfrak{A}_1 \otimes \mathfrak{B}_1$ i.e.,

$$\rho^{\otimes n} = \rho_{AB}^{\otimes n} \in \mathfrak{A} \otimes \mathfrak{B}$$

goal: create e-bits by applying sequence of

- local measurements with CC (classical communication)
- LO (local operations) conditioned on shared class alg

Entanglement measures

Entanglement of distillation: asymptotic rate $\dfrac{\# \text{ e-bits}}{\# \text{ copies of } \rho}$

$$\rho^{\otimes n} \mapsto (|\beta\rangle\langle\beta|)^{\otimes m}$$

Entanglement cost: asymptotic rate $\dfrac{\# \text{ e-bits}}{\# \text{ copies of } \rho}$

$$(|\beta\rangle\langle\beta|)^{\otimes m} \mapsto \rho^{\otimes n} \mapsto$$

LOCC not nec reversible: In general entang cost $>$ entang of dist

Entang cost $= \lim\limits_{n\to\infty} \mathrm{EoF}\big(\rho^{\otimes n}\big)$ \qquad Entanglement of Formation

$$\mathrm{EoF}(\rho) \equiv \sup\left\{ \sum_j p_j S\big(\mathrm{Tr}_B |\psi_j\rangle\langle\psi_j|\big) : \rho = \sum_j p_j |\psi_j\rangle\langle\psi_j| \right\}$$

Open question: NPT bound entanglement

Recall partial transpose $\mathcal{I} \otimes T$

PPT: state ρ_{AB} satisfies $(\mathcal{I} \otimes T)(\rho_{AB)} \geq 0$

for $d > 2$ can be separable or entangled

NPT: state ρ_{AB} for which $(\mathcal{I} \otimes T)(\rho_{AB} < 0$ always entangled

Thm: (Horodecki) If ρ_{AB} is PPT but not separable, then

no useful entanglement can be distilled

not even one e-bit or EPR pair – called bound entanglement

Question: Can at least one e-bit be distilled from every NPT state?

Or Are there NPT states which are "bound entangled" ?

Can reduce question to consideration of special states

$$a \sum_j |f_j \otimes f_j\rangle + b \sum_{j<k} |\phi_{jk}^+\rangle\langle\phi_{jk}^+| + c \sum_{j<k} |\phi_{jk}^-\rangle\langle\phi_{jk}^-|$$

$$|\phi_{jk}^\pm\rangle = \tfrac{1}{\sqrt{2}}\big(|f_j f_k\rangle \pm |f_k f_j\rangle\big)$$

Watrous showed that here are states $\rho = \rho_{AB}$ such that

- no entanglement can be distilled from $\rho^{\otimes n}$, but
- one e-bit can be distilled from $\rho^{\otimes(n+1)}$

Operator algebra reformulation

recall iso ρ_{AB} and CP map Υ give by Choi matrix

For ρ_{AB} NPT define $\Lambda = \Upsilon \circ T$

Claim: $\rho_{AB}^{\otimes n}$ is not distillable $\forall\ n$ $\quad \Leftrightarrow \quad$ $\Lambda^{\otimes m}$ is 2-poisitve $\forall\ m$

Challenge for Op Alg: Find a CP map Υ for which $(\Upsilon \circ T)^{\otimes m}$

$= (\Upsilon \circ T) \otimes (\Upsilon \circ T) \otimes \ldots \otimes (\Upsilon \circ T)$ is 2-positive for all m

OR show that no such map exists.

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