Mixed norms and entropy

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- ✎ T is state preserving if $f \geq 0$ and $\sum_k f(k) = 1$ implies $\sum_j T(f)(j) = 1$.
- ✎ A channel is given by a matrix $T(f)(j) = \sum_{jk} a_{jk} f_k$ such that $a_{jk} \geq 0$ and $\sum_j a_{jk} = 1$ for all j.

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Then $\sum_k f_k = 1$ implies

$$S(f) = -\frac{d}{dp}\|f\|_p\Big|_{p=1}\;.$$

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✏ Key observation

$$-S_{min}(T) = \max_{f} \frac{d}{dp} \| T(f) \|_p \Big|_{p=1} = \frac{d}{dp} \| T : \ell_1 \to \ell_p \| \Big|_{p=1} .$$

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Here the norm of a map $T : X \to Y$ is given by

$$\|T\| \;=\; \sup_{\|x\|\leq 1} \|Tx\|_Y\,.$$

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For $T(f)(j) = \sum_k a_{jk} f_k$ we have

$$(id \otimes T)((f_{ik}))_{i,k} \;=\; (\sum_k a_{jk} f_{ik})_{ij} \; .$$

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$$\|T \otimes S : \ell_p^{nm} \to \ell_q^{nm}\| \;=\; \|T\|\|S\| \, .$$

Proof: Let $\mathrm{flip} : \mathbb{C}^n \otimes \mathbb{C}^m \to \mathbb{C}^m \otimes \mathbb{C}^n$ be the flip map $\mathrm{flip}(x \otimes y) \;=\; y \otimes x$.

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$$\| \,\mathrm{flip}(id \otimes S) : \ell_p^{nm} \to \ell_q^m(\ell_p^n) \| \; \leq \; \| S \|,$$

and

$$\| \,\mathrm{flip}(id \otimes T) : \ell_q^m(\ell_p^n) \to \ell_q^{nm} \| \; \leq \; \| T \|.$$

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Hence $T \otimes S = \mathrm{flip}(id \otimes T)\,\mathrm{flip}(id \otimes S)$ satisfies

$$\|T \otimes S\| \;\le\; \|T\|\|S\| \,.$$

Application: Chain rule

Let S, T be channels

$$
\begin{aligned}
& -S_{\min}(S\otimes T) \\
& \frac{d}{dp}\|T\otimes S:\ell_1^{nm}\to\ell_p^{nm}\|\Big|_{p=1} \\
& = \frac{d}{dp}\|T:\ell_1^{n}\to\ell_p^{n}\|\Big|_{p=1}\|S:\ell_1^{m}\to\ell_p^{m}\| \\
& = \|T:\ell_1^{n}\to\ell_1^{n}\|\,\frac{d}{dp}\|S:\ell_1^{m}\to\ell_p^{m}\|\Big|_{p=1} \\
& + \frac{d}{dp}\|T:\ell_1^{n}\to\ell_p^{n}\|\Big|_{p=1}\|S:\ell_1^{m}\to\ell_1^{m}\| \\
& = -S_{\min}(T)-S_{\min}(S)\,.
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- ✍ and $||| \, id \otimes T : V_r(X) \to V_r(Y) \, ||| = ||| \, T \, |||$.

Theorem: Under the assumptions above

$$S_{\min}(T) \; = \; -\frac{d}{dp} \, ||| \, T : V_1 \to V_p \, |||$$

is additive on the class of maps satisfying $||| \, T : V_1 \to V_1 \, ||| = 1$.

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The minimal energy of Φ is given by

$$S_{min}(\Phi) = \inf_{tr(f)=1} S(\Phi(f)) .$$

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Conclusion: There is no family of norms on $L_p(M_n, tr) \otimes L_q(M_n, tr)$ satisfying Minkowski's inequality and

$$\|id \otimes \Phi : L_p(L_1) \to L_p(L_q)\| \ = \ \|\Phi\| \ .$$

Pisier's norm

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- (Theorem) For $T : L_{p_1}(M_n) \to L_{p_2}(M_n)$ the expression

$$|||T||| = \sup_m \|id \otimes T : L_q(M_m)[L_{p_1}(M_n)] \to L_q(M_m)[L_{p_2}(M_n)]\|$$

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- Interpolation, and connection with Haagerup tensor product.

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Warning: $S_{cb}(id_{M_n}) = -\ln n$.

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2. $$|||\, \Phi : L_1 \to L_p \,||| \;=\; \sup_{\sum_k \|h_k\|^2=1} \; \inf_{\|a\|_{2p'}=1} \| \sum_k a^{-1} e_{kj} a^{-1} \otimes \Phi(|h_k\rangle\langle h_j|)\|_p$$

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The first expression becomes

$$\omega_p(\Phi) \;=\; \sup_{\psi \in \mathbb{C}^n \otimes \mathbb{C}^n} \frac{\|id \otimes \Phi(|\psi\rangle\langle\psi|)\|_p}{\||(id \otimes tr)(\psi\rangle\langle\psi|)\|_p} \,.$$

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with the capacity of a channel for transmission of classical information with unlimited entanglement

$$C_{EA}(\Phi) = \sup_\psi (S(\gamma_1) + S(\gamma_2) - S(\gamma_{12}))$$

Connection to capacity

The first expression becomes

$$\omega_p(\Phi) \;=\; \sup_{\psi \in \mathbb{C}^n \otimes \mathbb{C}^n} \frac{\|id \otimes \Phi(|\psi\rangle\langle\psi|)\|_p}{\||(id \otimes tr)(\psi\rangle\langle\psi|)\|_p} \,.$$

For ψ and Φ fixed, let $\gamma_{12} = id \otimes \Phi(|\psi\rangle\langle\psi|)$, $\gamma_1 = (id \otimes tr)(\gamma_{12})$ and $\gamma_2 = (tr \otimes id)(\gamma_{12})$. Then we should compare

$$-S_{cb}(\Phi) = \sup_{\psi}(S(\gamma_1) - S(\gamma_{12}))$$

with the capacity of a channel for transmission of classical information with unlimited entanglement

$$C_{EA}(\Phi) = \sup_{\psi}(S(\gamma_1) + S(\gamma_2) - S(\gamma_{12}))$$

and the coherent information

$$C_Q(\Phi) = \sup_{\psi}(S(\gamma_2) - S(\gamma_{12}))$$

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(with Neufang and Ruan) For $\lambda(f) = \sum_g f(g)\lambda(g)$ and positive definite f with $f(e) = 1$ we have $S_{cb}(\Phi_f) \; = \; -\tau(\lambda(f) \ln \lambda(f))$

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a) Note that classical entropy is 0. b) The result also holds for finite quantum groups and should give new channels with good error correction properties.

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Theorem: (CL-2008) The expression

$$\|x\|_{L_p^+(L_q)} \;=\; \|(id \otimes tr(x^q))^{1/q}\|_p$$

is convex on positive matrices for $1 \leq p \leq q \leq 2$ or $q = 2$ and p arbitrary.

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Finally, $\|x\|_{L_p(L_q)} = \frac{1}{2}\| \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \|_{L_p(L_q)}$ for arbitrary x.

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- (with Xu) Let $1 < p < q < 2$. Then we have an inclusion into an asymmetric space

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Tricks: Tensor product. Central Limit theorem

$$\|\Phi(x^q)^{1/q}\|_p \ = \ \lim_n n^{-1/q} \|(\sum_{k=1}^{n} \pi_k(x)^q)^{1/q}\|$$

for suitable *-homomorphism constructed from the Krauss matrices for Φ.