Operator Spaces
and
Quantized Functional Analysis

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Toronto, July 2009

The theory of measurement begins with a pairing

$$\mathcal{O} \times \mathcal{S} \to \mathbb{R}$$

where $\mathcal{O}$ is the linear space of observables and $\mathcal{S}$ is the convex set of states. The standard model:

A a unital C*-algebra, $\mathcal{O} = A_{sa}$, $\mathcal{S} = S(A)$.

Kadison (1951): (1) The convex geometry of the state space $K = S(A)$ is determined by the order structure of A_{sa}.

(2) The (Jordan) algebraic structure of A_{sa}

$$a \circ b = \frac{1}{2}(ab + ba)$$

is determined by the geometry of K.

(3) $A_{sa} \cong \mathsf{Aff}(K)$

Kadison was using *classical functional analysis* (viz. Banach spaces and ordered Banach spaces) to study quantized systems. Specifically he sidestepped the quantized functional analysis (i.e., the underlying operator spaces and systems).

Although Kadison's program has been largely completed (Alfsen, Schutlz, Stormer) quantized functional analytic methods have had a more profound affect on the subject. Of course these techniques didn't appear for several more decades.

Virtually all of the invariants of operator algebra theory (e.g., nuclearity, exactness, local reflexivity, injectivity, etc.) are properties of this "hidden" linear structure.

We begin with the classical notions.

A *function space* is a closed complex subspace $V \subseteq \ell^\infty(\Omega)$.

If V is an arbitrary complex Banach space, it is isometric to a function system: $V \hookrightarrow \ell^\infty(\Omega)$ where Ω is the unit ball of the dual space V^*.

Abstractly a function space is just a Banach space $(V, \|\ \|)$. This category is closed under quotients, duals, mapping spaces, tensor products, etc.

A *function system* is a real closed linear subspace $V \subseteq \ell^\infty_{\mathbb{R}}(\Omega))$, where Ω is an arbitrary set and $1 \in V$. V is a *normed and ordered space*.

Abstractly, a function system $(V, \|\ \|, V^+, 1)$ is an ordered Banach space with a distinguished "unit" 1 such that $\|v\| \leq 1$ if and only if $-1 \leq v \leq 1$, $v \leq 0$ if and only if $v \leq \epsilon 1$ for all $\epsilon > 0$.

Examples of function systems

(1) $V = \mathbb{R}^n = \ell^\infty([n])$.

(2) Kadison: A a unital C*-algebra

$$V = A_{sa} = \mathsf{Aff}(S(A)) \subseteq C_{\mathbb{R}}(S(A))$$

(3) K compact convex set

$$V = \mathsf{Aff}(K) \subseteq \ell^\infty_{\mathbb{R}}(K)$$

$V^*, V/W$ are generally not function systems. E.g., $(\mathbb{R})^{n*} = \ell^1([n])$ is not a function system. However,

(4) Local function system: If $p \in S(V)$ let

$$\begin{aligned} W_p &= F_p - F_p \\ F_p &= \{q \in V^* : \exists c > 0 : 0 \leq q \leq cp\} \end{aligned}$$

with the relative order and the norm,

$$\|q\| = \min\{c : -cp \leq q \leq cp\}$$

is a function system with order unit p.

Morphisms:

$$\varphi : V \to W,\ \varphi \geq 0,\ \varphi(1) = 1$$

Examples:

(1) A *state* on V is a morphism $\varphi : V \to \mathbb{R}$.

(2) $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is a morphism if and only if $\psi = \varphi^* : \mathbb{R}^m \to \mathbb{R}^n$ is a channel.

(3) Duality theorems:

$$V \cong \mathsf{Aff}(S(V))$$

$$K \cong S(\mathsf{Aff}(S(V))$$

Function spaces $\cong$ compact convex sets

$$V \longrightarrow K = S(V)$$

$$K \longrightarrow V = \mathsf{Aff}(K)$$

This depends upon the Krein-Hahn-Banach theorem.

Def: Z is *injective* if given $1 \in W \subseteq V$, any morphism $\varphi : W \to Z$ has an extension to a morphism $\Phi : V \to Z$.

$$\begin{array}{ccc} V & & \\ \cup| & \searrow^{\exists\Phi} & \\ W & \xrightarrow{\varphi} & Z \end{array} \quad .$$

Krein-Hahn-Banach theorem: $Z = \mathbb{R}$ and $Z = \ell_\infty$ are injective.

V^{**} is injective if and only if V is a simplex space, i.e., $K = S(V)$ is a Choquet simplex.

Tensor products

Write V, V^d for vector spaces in duality.

$$\begin{aligned} V \otimes W &\simeq L(V^d, W) : u \mapsto \varphi_u \\ & \qquad \varphi_{v \otimes w}(f) = f(v)w \\ (V \otimes W)^d &\simeq L(V, W^d) : F \mapsto \varphi_F \\ & \qquad \varphi_F(v)(w) = F(v \otimes w) \end{aligned}$$

A cone $C \subseteq V$ is a convex subset such that $\mathbb{R}^+ C \subseteq C$. If $T \subseteq V$, cone(T) is the smallest cone containing T. Given linear spaces V and V^d in duality, each $T \subseteq V$ determines a dual cone $T^o = \{f \in V^d : f(T) \geq 0\}$ in V^d.

Given cones $C_1 \subseteq V_1$, and $C_2 \subseteq V_2$, we let

$$\begin{aligned} C_1 \odot C_2 &= \text{cone}(C_1 \times C_2) \\ C_1 \otimes C_2 &= (C_1^o \times C_2^o)^o. \end{aligned}$$

Given function spaces V_1 and V_2, V_1^+ and V_2^+, determine two tensor product function spaces $V_1 \otimes_{\mathsf{max}} V_2$ and $V_1 \otimes_{\mathsf{min}} V_2$. Given $K_i = S(V_i)$, we let

$$\begin{aligned} K_1 \mathbin{\Delta} K_2 &= S(V_1 \otimes_{\mathsf{min}} V_2) \\ K_1 \boxdot K_2 &= S(V_1 \otimes_{\mathsf{max}} V_2) \end{aligned}$$

Theorem (Kelley, Namioka, 1969): The following are equivalent for a fn system V

(1) $V \otimes_{\mathsf{min}} W = V \otimes_{\mathsf{max}} W$ for all fn systems W

(2) V^{**} is an injective function system

(3) $K = S(V)$ is a Choquet simplex

(4) $K \mathbin{\Delta} \square = K \boxdot \square$ (Riesz decomposition property for V^*).

Key observation: $p \in S(V \otimes_{\text{max}} W)$ determines a linear mapping $\varphi_p : V \to W^*$, with $p_0 = \varphi_p(1) \in S(W)$. $\varphi_p(V) \subseteq [p_0]$, hence $\varphi_p : V \to [p_0]$ is a morphism of function systems.

The quantized version of this classical convexity result led to the nuclearity/injectivity breakthrough in operator algebra theory.

(Lance,Choi,E) (See E "Injectives and tensor products, 1972" for the early history).

A decade later, the quantized version of normed spaces led to breakthroughs in the nonnuclear theory such as exactness, local reflexivity etc.

(Haagerup, E, Kirchberg)

Quantization: replace functions by operators and identify the natural mappings

function spaces $\rightsquigarrow$ operator spaces

$$V \subseteq \ell^\infty(\Omega) \rightsquigarrow V \subseteq \mathcal{B}(H)$$

bounded mappings $\rightsquigarrow$ completely bounded mappings.

function systems $\rightsquigarrow$ operator systems

$$1 \in V \subseteq \ell^\infty_{\mathbb{R}}(\Omega) \rightsquigarrow I \in V = V^* \subseteq \mathcal{B}(H)$$

positive unital mappings $\rightsquigarrow$ completely positive unital mappings

channels $\psi : \mathbb{R}^m \to \mathbb{R}^m \rightsquigarrow$ quantum channels $\psi : \mathbb{M}_m \to \mathbb{M}_n$

dual theory: state spaces as universal compact convex sets $\rightsquigarrow$ matrix state spaces as universal matrix convex sets.

The matricial norms and orderings

An n-tuple $f = (f_1, \ldots, f_n)$ of functions f_k on Ω is a function on $n\Omega = \Omega \sqcup \ldots \sqcup \Omega$

$$f(x, k) = f_k(x_k)$$

$V^n \subseteq \ell^\infty(n\Omega)$ a fn space (resp. system)

$$\|f\| = \max\{\|f_k\|\} \qquad (1)$$
$$f \geq 0 \Leftrightarrow f_k \geq 0 \qquad (2)$$

A matrix $v = \left[v_{ij}\right]$ of operators on H is an operator on H^n :

$$\begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \\ v_{n1} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = etc.$$

$M_n(V) \subseteq \mathcal{B}(H^n)$ is an op space (resp. system).

Key distinction: There are no matrix analogues of (1) and (2). The norm and ordering on $M_n(V)$ are not determined by the norm and/or order on V.

The matrix norms and orderings form an essential part of the structure of operator spaces and systems and must be acknowledged by the relevant mappings. $\varphi : V \to W$ determines

$$\varphi_n : M_n(V) \to M_n(W) : [v_{ij}] \to [\varphi(v_{ij})]$$

The completely bounded norm is given by

$$\|\varphi\|_{cb} = \sup\{\|\varphi\|_n\}$$

φ is *completely bounded* (resp. *completely contractive*) if $\|\varphi\|_{cb} < \infty$ (resp. ≤ 1). φ is a *complete isometry* if all of the mappings φn are isometric.

φ is *completely positive* if $\forall n, \varphi_n \geq 0$. If V, W are operator systems, a *morphism* $\varphi : V \to W$ is a unital completely positive mapping.

The quantum channels $\psi : M_m^* \to M_n^*$ are the adjoints of morphisms (completely positive unital maps) $\varphi : M_n \to M_m$.

Operator spaces are the quantized normed spaces.

Def: *Matrix normed (complex) vector space* vector space V with a distinguished Banach norm $\| \, \|_n$ on each of the matrix spaces $M_n(V)$.

Theorem (Ruan): A matrix normed space is completely isometric to an operator space if and only if

$$\|v \oplus w\| = \max\{\|v\|, \|w\|\}$$

and

$$\|\alpha v \beta\| \leq \|\alpha\| \, \|v\| \, \|\beta\|$$

where $v \oplus w$ is the diagonal sum, $\alpha v \beta$ is defined by matrix muliplication for $\alpha \in M_{m,n}, v \in M_n(V)$, $\beta \in M_{n,m}$.

Corollary: $V/W, V^*, \mathcal{CB}(V,W)$ are all operator spaces.

Note: A matrix of mappings is a mapping: if $\varphi_{i,j} : V \to W$, then we define $\varphi : V \to M_n(W)$ by

$$[\varphi_{i,j}](v) = [\varphi_{i,j}(v)]$$

and thus

$$M_n\mathcal{CB}(V, W) = \mathcal{CB}(V, M_n(W))$$

In particular,

$$M_n(V^*) = \mathcal{CB}(V, M_n)$$

The "history" of completely positive and completely bounded mappings.

Stinespring-GNS theorem (1955)

Arveson-Wittstock Hahn-Banach theorem for operator spaces and complete contractions (1969). Not all von Neumann algebras dual C^*-algebras are injective!

Lance characterization of $p \in S(A \otimes_{\max} B)$ in terms of completely positive mappings φ_p : $A \to B^*$ (1973) . This was the first indication that one must consider the matrix orderings on the dual space of a C^*-algebra.

Local op systems (Lance E 1977 (1973)) Let $p_0 = \varphi_p(1) \in S(B)$ and use the identification $[p_0] \cong \pi_0(B)'$. Led to semidiscrete von Neumann algebras.

This was an ingredient of Connes' characterization of the injective von Neumann algebras as just the hyperfinite algebras (1976)

A C*-algebra A is nuclear if and only if A^{**} is injective (Choi E Kirchberg 1977)

Also the characterization of the nuclear C^*-algebras by approximately commuting diagrams of morphisms

$$\begin{array}{ccccc} & & M_n & & \\ & \varphi \nearrow & & \searrow \psi & \\ A & & \xrightarrow{id} & & A \end{array}$$

Haagerup discovered (1978) the existence of finite rank complete contractions and finite dimensional operator spaces V_n

$$\begin{array}{ccccc} & & V_n & & \\ & \varphi \nearrow & & \searrow \psi & \\ C^*_{\mathrm{reg}}(\mathbb{F}_n) & & \xrightarrow{id} & & C^*_{\mathrm{reg}}(\mathbb{F}_n) \end{array}$$

This more general approximation property made it clear that the underlying operator space structure is important for the algebraic classification of non-nuclear C^*-algebras.

Exactness, local reflexivity, etc. These phenomena are purely non-classical!

The liberation of quantum functional analysis from operator algebra theory.

Just as Banach spaces are ubiquitous in analysis, the same applies to operator spaces in "quantized analysis". Completely bounded mappings play a key role in the harmonic analysis on non-commutative groups, the theory of non-commutative boundaries (see Arveson) and non-self-adjoint operator algebras (Arveson, Blecher, Paulsen, etc)

Non-commutative L^p-spaces, interpolation theory (Pisier, Xu).

All those Banach space tensor products and mapping spaces (the Grothendieck program and the Grothendieck theorem) (Ruan, Junge, Pisier, Shlyakhtenko, E)

A surprising non-commutative example: If R is any von Neumann algebra, its predual $L^1(R) = R_*$ is always locally reflexive (Junge, Ruan, E)

Any Banach space E has (many) quantizations.

$$E \hookrightarrow \ell^\infty(\Omega) \hookrightarrow \mathcal{B}(\ell^2(\Omega))$$

determines $V = \text{MIN}(E)$, and $E \mapsto \text{MIN}(E)$ is a functor. Also one has a maximal quantization functor $\text{MAX}(E)$ (Blecher, Paulsen). For example,

E.g., let

$$\begin{aligned} Z_n &= \mathbb{C}z_1 + \ldots + \mathbb{C}z_n \subseteq C^*(\mathbb{Z}^n) = C(\mathbb{T}^n) \\ E_n &= \mathbb{C}u_1 + \ldots + \mathbb{C}u_n \subseteq C^*(\mathbb{F}_n) \end{aligned}$$

Then $\text{MIN}(\ell^1_n) \cong Z_n$ and $\text{MAX}(\ell^1_n) \cong E_n$.

Quantum information theory and the return of operator systems (Paulsen-Todorov-Tomforde).

Strictly speaking, the generalization of the Kelley-Namioka theory (simplex spaces $\rightsquigarrow$ nuclear C^*-algebras) is incomplete since it considered the analogy between function systems and unital C^*-algebras. A more satisfactory theory would be to consider general operator systems.

Up until recently such a theory seemed to be of only academic interest. That has suddenly changed.

PTT show that there are natural functors OMIN and OMAX from function systems to operator systems with the expected properties.

Of particular interest is that a mapping $\varphi : M_n \to M_m$ is entanglement breaking if and only if the corresponding mapping

$$\varphi : \text{OMIN}M_n \to \text{OMAX}M_m$$

is an operator space morphism, i.e., it is completely positive.

It is evident that the next step is to find a complete (forgive the pun) analogue of the remarkable Kelley-Namioka theorem. To begin with it is clear how one should define the tensor products. This can be done by using the appropriate matrix convex cones $C_n^{\odot}$ and $C_n^{\otimes}$ determined by the matrix positive cones of V_1 and V_2 and their duals. Matrix convexity is yet another quantized notion which has already had some impact on QIT, but cannot be discussed in this lecture.

It would seem likely that the local theory will shed some light on the elegant purification techniques of QIT and conversely.

Thank you for your attention.