

Geometric properties of the multivariate Bernstein basis and certificates of positivity

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1 Introduction

This talk: report on Richard Leroy's Ph D thesis in Rennes on the multivariate case after my joint work with Fatima Boudaoud and Fabrizio Caruso in the univariate case.

Certificates

If a polynomial is positive, a certificate of positivity is a way of writing down the polynomial making clear that it is positive.

In other words, an answer by yes or no is not satisfactory, a certificate is needed.

Example of certificate: extended euclidean algorithm guarantees the gcd by computing also cofactors.

Important in certified software.

Various possibilities for certificates of positivity

$$P(X_1, \dots, X_k) \in \mathbb{Z}[X_1, \dots, X_k]$$

Hilbert's 17 th problem:

is a polynomials everywhere positive a sums of squares ?

- not of polynomials (explicit counter example by Motzkin)
- but of rational functions (Artin and Schreier)
- very unexplicit proof (based on Zorn's lemma)
- valid in a general real closed field (not necessarily archimedean)
- gives no method to produce the certificate
- very hard problems, work in progress (Lombardi and Roy ...)

Polya's theorem:

certificate of positivity for a polynomial positive on an orthant

- after multiplying by $1 + X_1 + \dots + X_k$ enough times all the coefficients become positive
- gives a method
- valid only in the fields of reals (archimedeanity needed)
- very high degrees

Bernstein basis approach (this talk)

- *certificate of positivity on a simplex*
- more flexible (can change simplex)
- all Bernstein coefficients “become positive” (explanation in the talk)
- global approach: equivalent to Polya
- new more efficient local approach (recent)

Bernstein basis is used

- in approximation theory (classical)
- (univariate) in probability theory (classical)
- (multivariate) in computer aided design (modern)
- (univariate) in real root isolation (recent... Mourrain ...)
- (univariate) in certificates of positivity (recent Boudaoud, Caruso, Roy)
- (multivariate) in certificates of positivity (new Leroy)

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2 Multivariate Bernstein basis

Simplex $V = [v_0, \dots, v_k]$ defined by $k + 1$ linear inequalities, $X = (X_1, \dots, X_k)$ $\ell_j(X) \geq 0$, $j = 0, \dots, k$, normalized by

$$1 = \ell_0(X) + \dots + \ell_k(X)$$

The ℓ_i correspond to barycentric coordinates: a point inside V has positive barycentric coordinates between 0 and 1.

Multi-index $i = (i_0, \dots, i_k)$, of degree d , $|i| = i_0 + \dots + i_k = d$,

$$\binom{d}{i} = \frac{d!}{i_0! \dots i_k!} \text{ multinomial coefficient}$$

$$\ell^i(X) = \prod_{j=0}^k \ell_j^{i_j}(X)$$

Bernstein polynomial of multi-index i

$$\text{Bern}_{d,i}(V)(X) = \binom{d}{i} \ell^i(X) \tag{1}$$

Think of

$$1 = (\ell_0(X) + \dots + \ell_k(X))^d$$

Example

- a) $k = 1, d = 2, \ell_0 = 1 - X, \ell_1 = X$
 multi-indices $(2,0), (1,1), (0,2)$
 develop $((1 - X) + X)^2$ and collect the pieces

$$(1 - X)^2, 2 X (1 - X), X^2$$

- b) $k = 2, d = 2, \ell_0 = 1 - X - Y, \ell_1 = X, \ell_2 = Y$
 multi-indices $(2,0,0), (1,1,0), (1,0,1), (0,2,0), (0,1,1), (0,0,2)$,
 develop $((1 - X - Y) + X + Y)^2$ and collect the pieces

$$(1 - X - Y)^2, 2 X (1 - X - Y), 2 Y (1 - X - Y), X^2, 2 X Y, Y^2$$

- c) $k = 3, d = 2, \ell_0 = 1 - X - Y - Z, \ell_1 = X, \ell_2 = Y, \ell_3 = Z$
 develop $((1 - X - Y - Z) + X + Y + Z)^2$ and collect the pieces

Properties of the Bernstein basis

- takes positive values on V ,
- basis of the vector-space of polynomials of degree $\leq d$
- 1 has coefficients $(1, \dots, 1)$

If $\deg(P) \leq d$ and i a multiindex of degree d , denote by $b(P, d, V)_i$ (or simply b_i) the coefficient of $\text{Bern}_{d,i}(V)$ in P and by $b(P, d, V)$ the vector of Bernstein coefficients of P .

The values of P at the vertices of the simplex are given by $b(P, d, V)_{de_j}$, $e_j = (0, \dots, 0, 1, 0, \dots, 0)$, 1 at place j , $j = 0, \dots, k$:
at a vertex all barycentric coordinates are equal to 0 but one of them.

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3 How to define the control polytope

Grid points in the simplex V in degree d :

to a multiindex $i = (i_0, \dots, i_k)$ such that $d = i_0 + \dots + i_k$ is associated a grid point m_i with barycentric coordinates $(i_0/d, \dots, i_k/d)$

Control points of \mathbf{P} : points $M_i = (m_i, b_i)$ where $b_i = b(P, d, V)_i$

In dimension one, the control points define the *control line*.

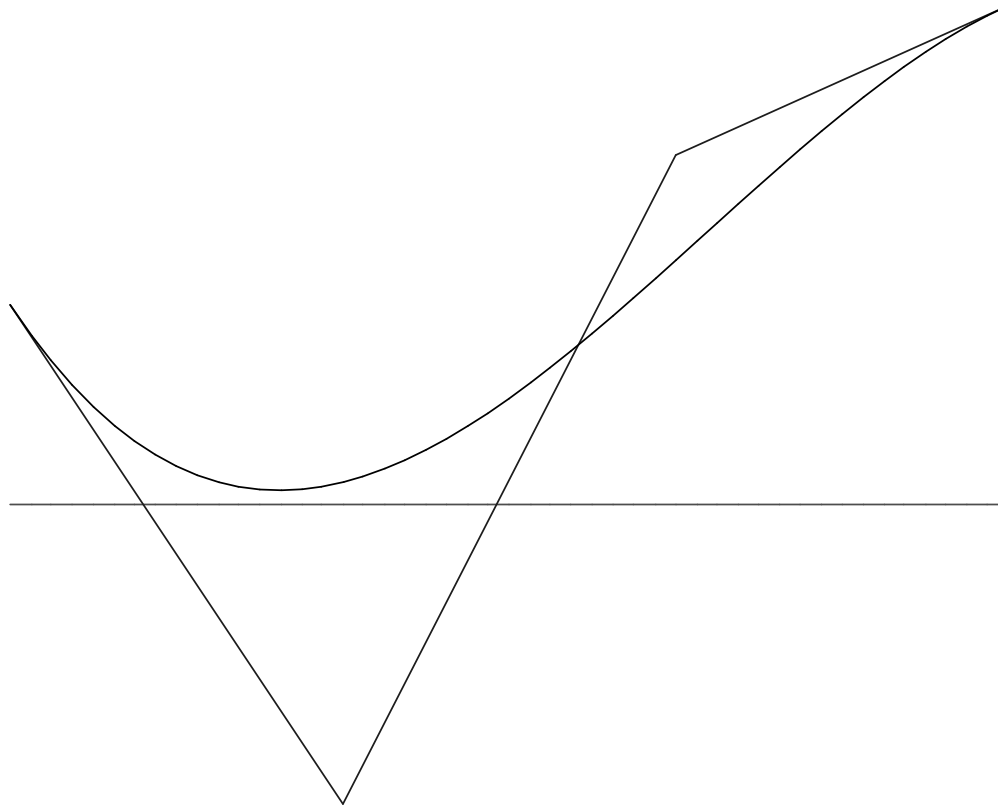


Figure 1.

In dimension more than one, the control points do not define a *control polytope* above V . It is needed to define first a triangulation of V based on the grid points m_i .

Example in dimension 2: we need to choose how to triangulate the square.

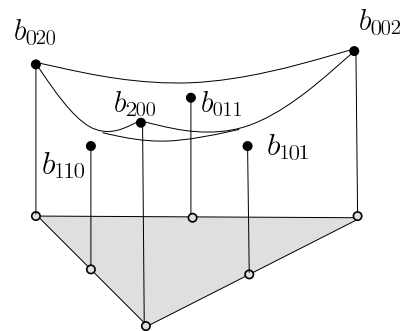


Figure 2.

1 Standard triangulation [GP]

Let V be a simplex with affinely independent vertices v_0, \dots, v_k , d the degree. The definition of the standard triangulation $T_{k,d}(V)$ is not intrinsic and depends on the order of the vertices of V .

For simplicity we describe the case of the unit simplex Δ , the general case being an affine transformation of this special case.

Kuhn's triangulation of the unit cube

To a permutation σ of $1, \dots, k$ is associated a simplex $V_\sigma = [v_{\sigma,0}, \dots, v_{\sigma,k}]$ defined by

- $v_{\sigma,0} = (0, \dots, 0)$
- $v_{\sigma,i} = e_{\sigma(1)} + \dots + e_{\sigma(i)}$ for $i = 1, \dots, k$

where e_i is the canonical basis.

The collection of these simplices triangulate the unit cube.

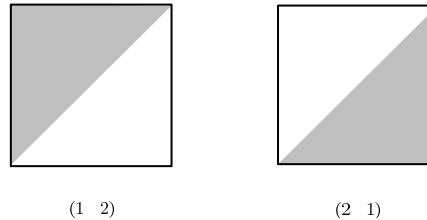


Figure 3.

Figure in dimension 3.

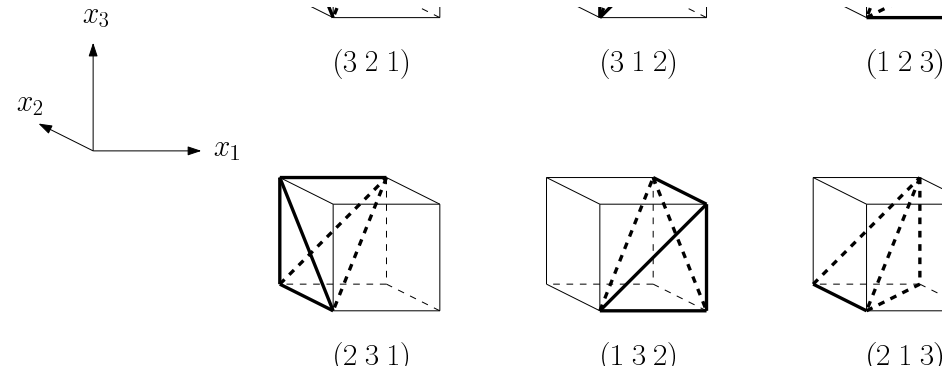


Figure 4.

Adjacency relations between two simplices are easy to describe :

V_σ and V_τ are adjacent if and only if there exists i such that $\sigma(i) = \tau(i + 1)$, $\sigma(i + 1) = \tau(i)$.

Standard triangulation (unit simplex Δ)

Send the unit cube equipped with Kuhn's triangulation to a distorted cube inside the unit simplex by $e_1 \rightarrow e_1/d, e_2 \rightarrow (e_2 - e_1)/d, \dots, e_k \rightarrow (e_k - e_{k-1})/d$

Translate these “cubes” in the direction of the axis and observe that the restriction to the unit simplex Δ of the Kuhn triangulations of these cubes is a triangulation of Δ .

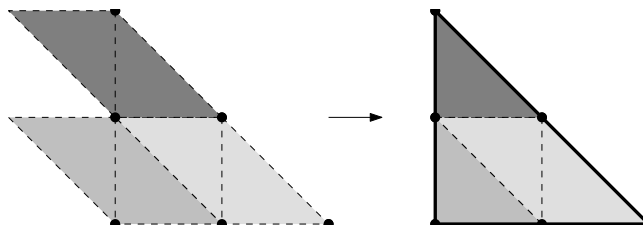


Figure 5.

Figure in dimension 3.

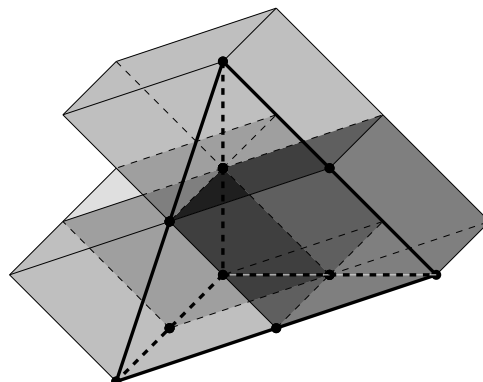


Figure 6.

Properties of the standard triangulation

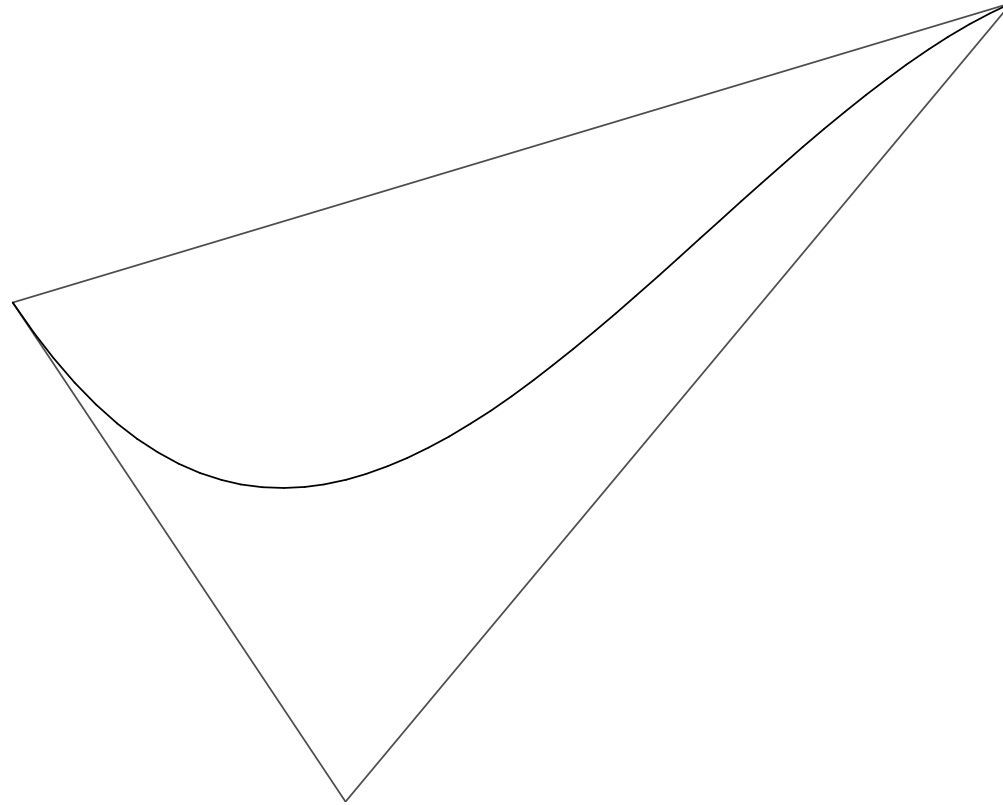
- it has a nice combinatorial description in terms of mappings from $\{1, \dots, k\}$ to $\{1, \dots, d\}$ (not explained here, see Richard's thesis)
- adjacencies between simplices have a nice combinatorial description
- it depends on the order of the vertices but it is invariant under a cyclic permutation of the vertices
- its vertices are grid points
- the restriction of $T_{k,d}(\Delta)$ to the simplex V with vertices v_0, \dots, v_r is $T_{r,d}(V)$
- if V is a simplex of $T_{k,d}(\Delta)$, $T_{k,\ell}(V)$ is the restriction to V of $T_{k,d\ell}(\Delta)$

Control polytope

Once the standard triangulation $T_{k,d}(V)$ of V is defined, it makes sense to define the control polytope of a polynomial P on V associated to the standard triangulation: it is the piecewise linear continuous function defined over each $W = [v^0, \dots, v^k]$ of $T_{k,d}(V)$ by the corresponding control points.

The control polytope of P associated to the standard triangulation is a kind of piecewise linear approximation of the graph of P .

The graph of P on V is contained in the convex hull of the control polytope.

**Figure 7.**

Convexity

Adjacency relations between sub-simplexes of $T_{k,d}(V)$ are easy to describe in a combinatorial mannner.

As a consequence, the control polytope of P on V is convex if and only if, with $e_j = (0, \dots, 0, 1, 0, \dots, 0)$, and $e_{-1} = e_k$

$$b_{i+e_j+e_{\ell-1}} + b_{i+e_{j-1}+e_\ell} \geq b_{i+e_{j-1}+e_{\ell-1}} + b_{i+e_j+e_\ell}$$

for all $0 \leq j < \ell \leq k$ and all multi-index i of degree $d - 2$.

To P is associated the vector $\delta_2(b)$ whose i, j, ℓ 's coordinate is

$$b_{i+e_j+e_{\ell-1}} + b_{i+e_{j-1}+e_\ell} - b_{i+e_{j-1}+e_{\ell-1}} - b_{i+e_j+e_\ell}.$$

It can be interpreted as Bernstein coefficients of second order derivatives in specific directions (convexity is related to second order derivatives !).

Example $k = 2, d = 2$ the vector $\delta_2(b)$ has three components

$$b_{(2,0,0)} + b_{(0,1,1)} - b_{(1,1,0)} - b_{(1,0,1)}$$

$$b_{(0,2,0)} + b_{(1,0,1)} - b_{(1,1,0)} - b_{(0,1,1)}$$

$$b_{(0,0,2)} + b_{(1,1,0)} - b_{(1,0,1)} - b_{(0,1,1)}$$

See Figure 2.

2 Worse possible distance between the graph and the control polytope for the standard simplex

Theorem 1. *The maximum distance between the graph of P and the control polytope of P on the standard simplex Δ is estimated by*

$$\frac{d k (k + 2)}{24} \|\delta_2(b)\|_\infty$$

When $k = 1$, one recovers the classical bound

$$\frac{d}{8} \|\delta_2(b)\|_\infty$$

When $k = 2$, one recovers the bound from [Re]

$$\frac{d}{3} \|\delta_2(b)\|_\infty$$

Idea of the proof:

- use convexity and prove that, supposing without loss of generality $\|\delta_2(b)\|_\infty = 1$, the maximum distance is obtained for a polynomial P such that all components of $\delta_2(P)$ are 1
- construct explicitly a polynomial P^* such that all components of $\delta_2(b^*)$ are 1 and compute the difference between the graph and the control polytope for P^*

It turns out that there is a polynomial P^* of degree 2 such that $\delta_2(b^*) = 1$. It is the quadratic form associated to the symmetric matrix

$$m_{i,j} = \frac{d(d-1)}{2} i(k-j+1), i \leq j$$

If $k = 2$ and $d = 2$, we obtain

$$2X^2 + 2XY + 2Y^2$$

which was known to reach the maximum.

For $k > 2$, the result is new.

As a consequence we get a (new) result explaining how the values at the grid points are approximated by the Bernstein coefficients (note that the control polytope does not appear anymore in the statement, it is used only in the proof)

Corollary 2. *The maximum distance between the values of P at the grid points and the Bernstein coefficients of P on the standard simplex Δ is estimated by*

$$\frac{d k (k + 2)}{24} \|\delta_2(b)\|_\infty$$

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4 Certificates of positivity on a simplex

Suppose that P is positive on Δ . By a certificate of positivity we mean an algebraic identity proving that P is positive on Δ . There are two kinds of certificates of positivity in the Bernstein basis:

Global certificates of positivity (classical)

Express P in the Bernstein basis for increasing degree D . If D is big enough, all the coefficients are positive (proved by Bernstein).

Local certificates of positivity (new)

Keep the degree d and subdivide Δ in subsimplices for which all the coefficients of P are positive.

We denote by m the minimum of P on Δ .

Global certificates of positivity

Express P in the Bernstein basis for increasing degree D . If D is big enough, all the coefficients are positive.

Theorem 3. *If P is positive on Δ*

$$D > \frac{d(d-1)k(k+2)}{24m} \|\delta_2(b)\|_\infty$$

ensures that all the elements of $b(P, D, \Delta)$ are positive.

Note: already existing result by [PR], better in some cases, worse in some cases.

Local certificates of positivity

Keep the degree d and subdivide Δ in subsimplices for which all the coefficients of P are positive.

Theorem 4. *If P is positive on Δ*

$$2^N > \frac{\sqrt{d}k(k+2)\sqrt{k(k+1)(k+3)}}{24\sqrt{m}}\sqrt{\|\delta_2(b)\|_\infty}$$

ensures that all the elements of $b(P, D, V_i)$ are positive for V_i a simplex of the standard triangulation $T_{2^N}(\Delta)$.

Local certificates are better for two reasons

- the size of the certificates is smaller: \sqrt{m} rather than m at the denominator
- the process is adaptative, since some simplices do not need to be subdivided.

Note that the behaviour with respect to k is worse.

The only remaining question is: if P is not everywhere positive, how to be sure that the algorithm stops ? This is done through estimating the minimum of a multivariate polynomial [JP] (improving [BLR]).

Rough description of the Certificate of Positivity Algorithm

Initialize the list L of simplices to inspect with Δ

Remove a simplex V from L

If all the elements of $b(P, d, V)$ are positive, store them in a list C

If a value of P at a vertex of V is negative output it

Otherwise subdivide V using the standard triangulation of degree 2, and put all the simplices of $T_2(V)$ in L .

Stop when L is empty OR the diameter of each V on L is small enough to ensure that P is not everywhere positive.

Examples

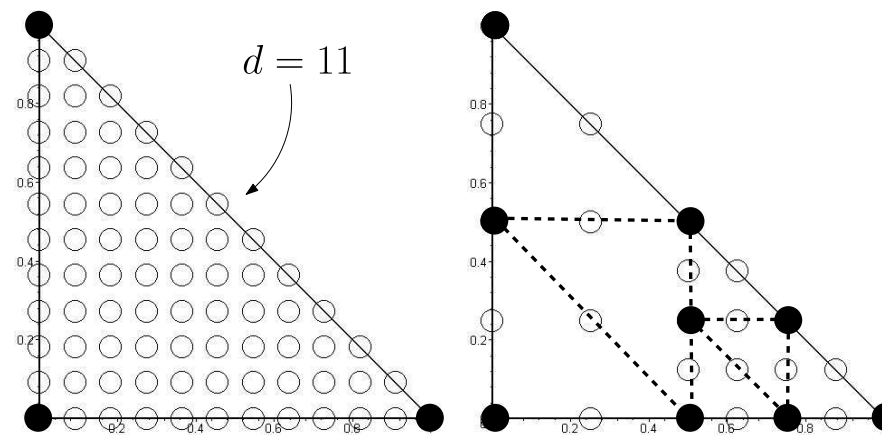


Figure 8.

5 Conclusions and future work

In the univariate case, the consideration of local certificates led to polynomial size certificates of positivity (in the degree and bitsize of integer coefficients) [BCaR] while the global approach was exponential (explicit exponential examples). Key tool: the number of variations in the list of Bernstein coefficients is subadditive: after a subdivision in two segments the sum does not increase.

In the multivariate case we do not have any similar discrete quantity which measures the number of sign variations before and after the subdivision ...

So, in the multivariate case, local certificates are better than global certificates, a good practical behaviour can be observed but we do not know whether the complexity is significantly improved.

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