Polar and Bipolar Varieties

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Workshop on Complexity of Numerical Computation

REAL SOLVING [Bank–G–Heintz–Lehmann–Pardo]

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If the system of those equations has real solutions at all, the solution set can consist of a finite number of points or can be a variety of positive dimension.

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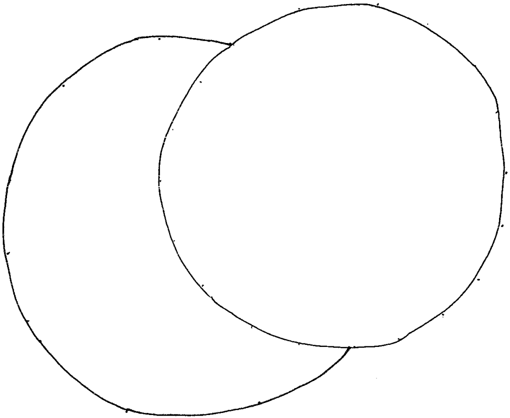
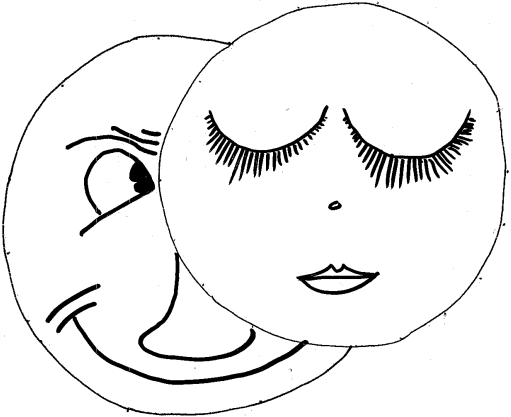
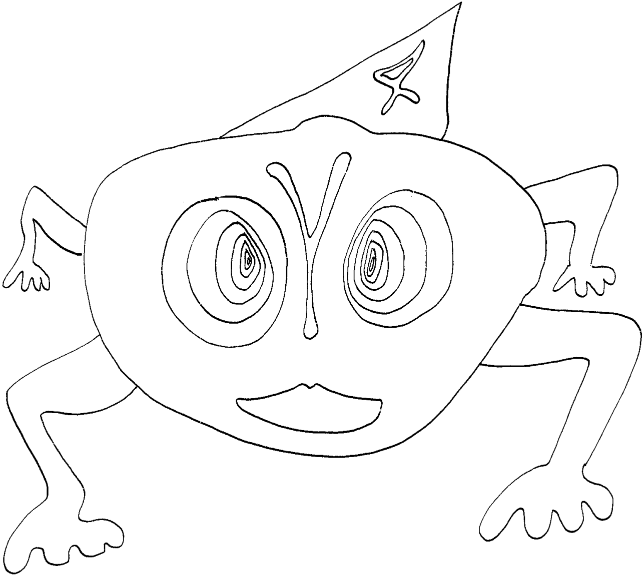
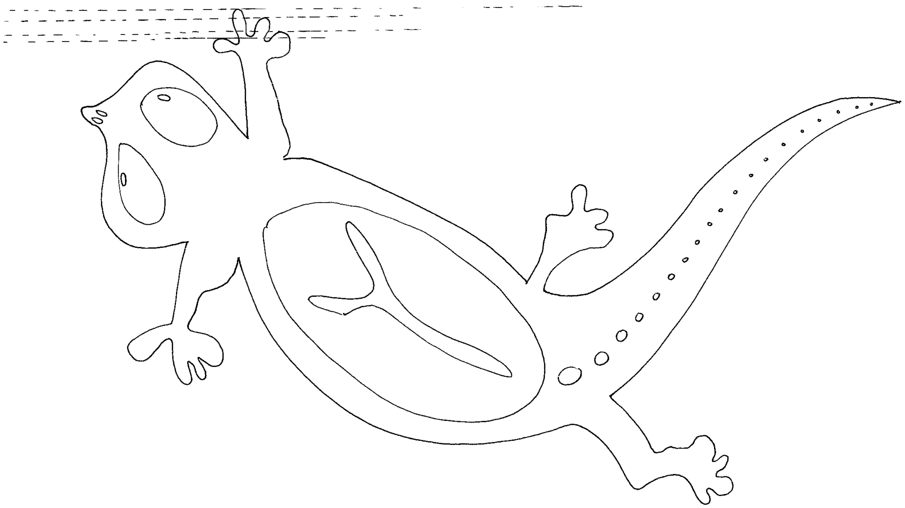
It turns out that the algorithm of the TERA–project performes very well with this task and is able to solve a larger number of examples than the best known commercial polynomial solvers.

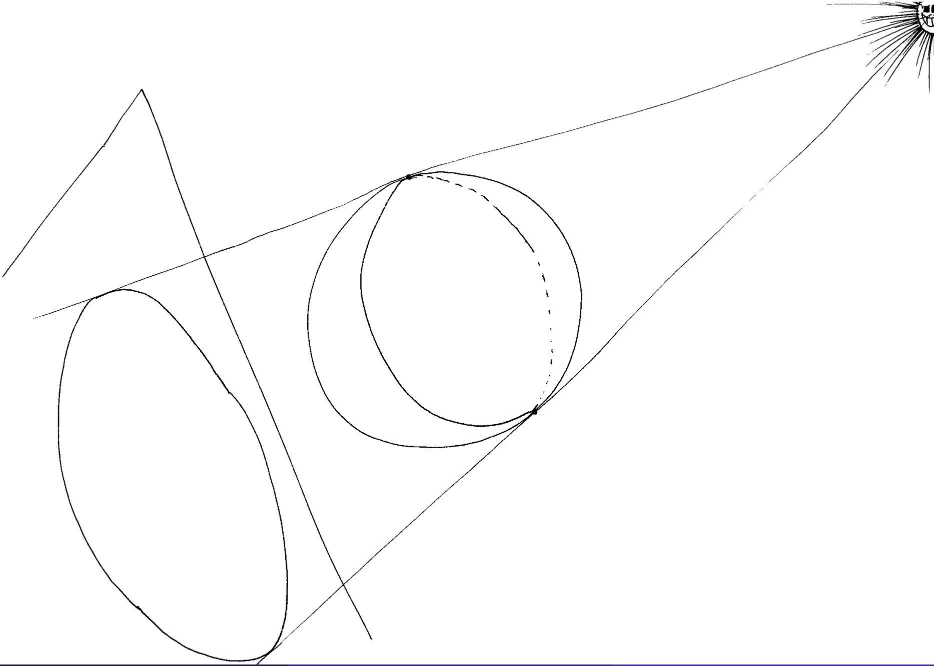






An algebraic curve





Geometric resolution

- Late eighties ... early nineties seminal historical papers by [G–Heintz], [G–Heintz–Pardo et al.], ... [Krick–Pardo]
- Design of algorithms [G–Lecerf–Salvy], [Durvye–Lecerf], [Durvye '08], [Heintz–Krick–Puddu–Sabia–Waissbein]
- Implementation [Lecerf], Lehmann, Schost: KRONECKER
- TERA group
  Aldaz, Avendaño, Bank, Beltran, Bostan, Cafure, Castaño, Castro, Dickenstein, Durvye, Fitchas, G, Grimson, Hägele, Heintz, Jeronimo, Krick, Lecerf, Lehmann, Llovet, Mandel, Marchand, Matera, M'bakop, Montaña, Morais, Morgenstern, Pardo, Puddu, Sabia, Salvy, Schost, Sedoglavic, Sessa, Smietanski, Solernó, Turull, Wachenchauzer, Waissbein, ...

Introduction

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  intrinsic complexity.
- In particular to find efficiently one representative point in every connected component of a real algebraic variety constitutes a fundamental problem in real algebraic geometry. We tackle this problem by exploiting the geometric solving of generic or (sufficiently generic) polar varieties (by the algorithm `Kronecker`)
- Several generalisations of the notion of polar variety allows us to drop successively the assumptions of hypersurface, compactness and smoothness.

Polar varieties: Notations

Let A be a **generic** $((n-1)\times n)$–matrix.

For an index i, fixed between 1 and $n-1$, we denote by

$$A_i := \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n-i,1} & a_{n-i,2} & \cdots & a_{n-i,n} \end{bmatrix},$$

the sub–matrix of A formed by the lines $1, 2, \dots, n-i$ of A.

- $f \in \mathbb{Q}[x_1, \dots, x_n]$
- $\mathcal{V} := \mathcal{V}_{\mathbb{C}}(f) := \{x \in \mathbb{C}^n \mid f(x) = 0\}$
- $\mathcal{V}_{\mathbb{R}}$ its real trace $\mathcal{V}_{\mathbb{C}} \cap \mathbb{A}_{\mathbb{R}}{}^n$

Polar varieties: Definitions

The i–th (open) polar variety of $\mathcal{V}_{\mathbb{R}}$

is defined by

$$\Delta_i(A) := \{x \in \mathcal{V}_{\mathbb{R}} \setminus Sing\ \mathcal{V}_{\mathbb{R}} \mid T_x\ \mathcal{V}_{\mathbb{R}} \text{ not transversal to } \ker A_i\}.$$

$\Delta_i(A)$ is the set of all points in $\mathcal{V}_{\mathbb{R}}$,
where the $((1+n-i)\times n)$–matrix

$$\mathcal{A}_i(x) := \begin{bmatrix} \frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n} \\ A_i \end{bmatrix} \text{ is not of maximal rank, i.e.,}$$

$$rank\ \mathcal{A}_i(x) \leq n - i.$$

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The i–th (closed) polar variety of $\mathcal{V}_{\mathbb{R}}$

is defined as the Zariski–closure of the i–th open polar variety.

Polar varieties: Determinantal Equations in the Smooth Case

- If the hypersurface $\mathcal{V}_{\mathbb{R}}$ is smooth, one obtains natural equations by adjoining to $f = 0$ all maximal minors of the matrix $\mathcal{A}_i(x)$.
- According to the maximal codimension of determinantal varieties, the expected codimension of the i–th polar variety is i,

$$n - (1 + n - i) + 1 = i \ .$$

- Corresponding to the flag

$$\ker A_{n-1} \subset \ldots \subset \ker A_1 = \ker A$$

the polar varieties form a decreasing sequence of subvarieties of $\mathcal{V}$ with expected codimension $1, \ldots, n-1$.

Notations and problem

- $f \in \mathbb{Q}[x_1, \ldots, x_n]$
- $\mathcal{V} := \mathcal{V}_{\mathbb{C}}(f) := \{x \in \mathbb{C}^n \mid f(x) = 0\}$
- $\mathcal{V}_{\mathbb{R}}$ its real trace $\mathcal{V}_{\mathbb{C}} \cap \mathbb{A}^n{}_{\mathbb{R}}$
- The equation of the hypersurface $f = 0$ is assumed to be regular, i.e. : $\mathcal{V}_{\mathbb{R}}$ is non empty and $grad(f)$ does not vanish on any of its connected component.
- Degree of $f \leq d$

FIND AT LEAST A POINT IN EVERY CONNECTED COMPONENT OF $\mathcal{V}_{\mathbb{R}}$

Small history of the problem

- Grigoriev, Grigoriev/Vorobjov '87 , '88
- Complexity $d^{O(n)}$:
  Heintz/Roy/Solerno '89, '90
  Barvinok '91
  Renegar '92, ,'98
  Canny/Emiris, Canny '95, '98
  Blum/Cucker/Shub/Smale '97
  Cucker/Smale '98
  Basu/Pollack/Roy '95, '98
- More recent papers:
  Rouillier/Roy/Safey el Din
  Aubry/Rouillier/Safey el Din
  Safey el Din/Schost, Safey el Din
  ...
- Our contributions '97, '98, '00, '02, '05, '07, '08, '09

Polar Varieties: Quantification of Data Complexity

- $f \in \mathbb{Q}[x_1, \dots x_n]$ regular equation of a real smooth hypersurface

$$\mathcal{V}_{\mathbb{R}} = f^{-1}(0) \subset \mathbb{R}^n, \ \ \mathcal{V}_{\mathbb{R}}$$

- $\mathcal{V}_{\mathbb{R}}$ compact for the moment
- Degree of $f \leq d$
- Evaluation complexity of $f \leq L$
- Maximal **degree of the complex polar varieties** δ
- Always holds $\delta \ \leq d^n$, d^n is the **Bézout** number)

The smooth past: find a point in every connected component of $\mathcal{V}_{\mathbb{R}}$ when compact and smooth

: Complexity Theorem

- **Intrinsic Complexity** (number of arithmetic operations in $\mathbb{Q}$)

$$L(nd\delta)^{O(1)}$$

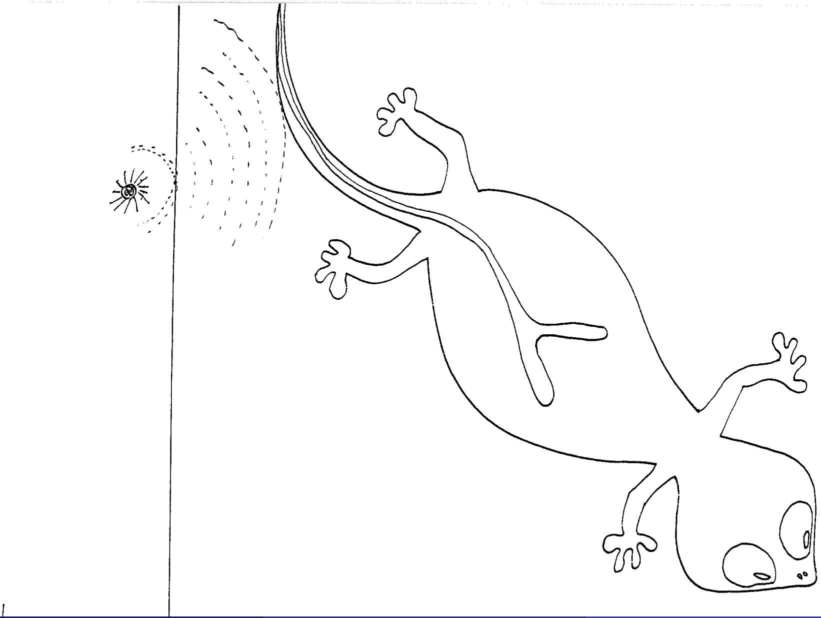
- Existence theorem (deterministic in a non–uniform setting)
- Probabilistic version (implementation)
- **Extrinsic complexity:**

Linear in $\mathbf{L}$

Polynomial in the Bézout number $\mathbf{d^n}$

We meet the best known extrinsic complexity bounds.

A non compact algebraic curve, still smooth



The non compact case

We consider a larger parameter space, the product of

- a flag variety,
- a variety of hyperquadrics
- a variety of hyperplanes.

The RADAR technology.

Again we arrive in the same complexity class!

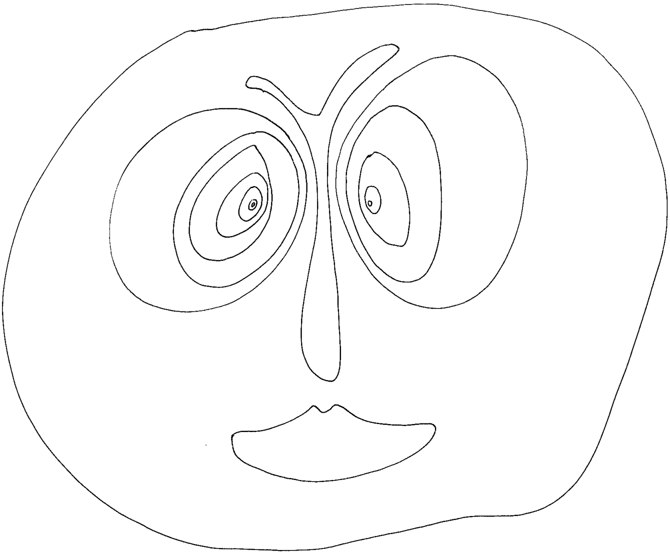
Complete intersection case; still smooth

- $f_1, \ldots, f_p$ a reduced regular sequence
- $\mathcal{V} \ := \{f_1 = \cdots = f_p = 0\}$
- $\mathcal{V}_{\mathbb{R}}$ is still smooth

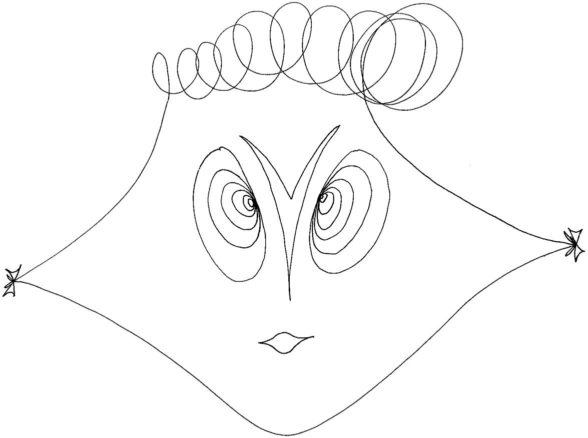
Up to a combinatorial factor we obtain the

same complexity class.

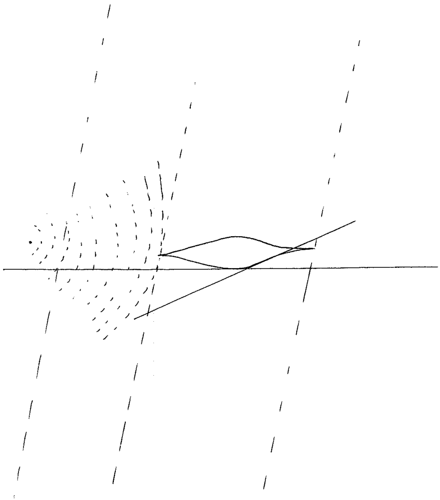
Again the old algebraic curve!



A new fresh one!!



Thom's lips



The hat of the bishop of Jerusalem

$$f(x, y, z) := z^2 - \varepsilon^2(x^2 + y^2 - 1)^3 = 0, \quad \varepsilon \quad \text{small}$$

The singular hypersurface case

- Hypersurface $\{f = 0\}$
- The $((1 + n - i) \times n)$–matrix

$$\mathcal{A}_i(x) := \begin{bmatrix} J(f) \\ A_i \end{bmatrix} \quad \text{is not of maximal rank}$$

  i.e. rank $\mathcal{A}_i(x) \leq n - i$
- $J(f)(x)$ denotes the gradient of f at $x := (\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n})$

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. . . The determinantal description

defined above by the maximal minors is no longer valid since it contains all the singular locus.

LISSIFICATION AND STRATIFICATION VERSUS DESINGULARIZATION

Canonical desingularization of determinantal varieties – à la ROOM–KEMPF

- In the following let i be a fixed index between 1 and $n-1$. We consider the linear system

$$J(f)(x)^T\lambda + {A_i}^T\mu = 0$$

- with x in $\mathcal{V}$ and (λ, μ) in $\mathbb{A}^1 \times \mathbb{A}^{n-i}$.

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Fundamental property: x singular implies $\mu = 0$

The singular case

The parameter space E_i

$$E_i := \{(x, A_i, \lambda, \mu) \in \mathbb{A}^n \times \mathbb{A}^{n(n-i)} \times \mathbb{A}^1 \times \mathbb{A}^{n-i} \,|\, rank\, A_i = n - i,\ \ \mu \neq 0\}$$

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Group action

On $E := E_i$ there is a natural right action of the group
$G := Gl(n-i) \times Gl(1)$:
Let $g := (B, t) \in G$ and $e := (x, A, \lambda, \mu) \in E$

$$E \times G \to E, \qquad (e, g) \mapsto e \cdot g := (x, B^T \cdot A, t\lambda, tB^{-1} \cdot \mu)).$$

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Quotient space

G induces on E_i an equivalence relation $\sim$.
We denote by $E_{/\sim}$ the set of G–orbits.

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- The points in a typical chart are of the form

$$A = \left[I_{n-i} \ \tilde{A} \right], \quad \mu = (1, \tilde{\mu})$$

  where I_{n-i} is the identity matrix, $\tilde{A}$ a $(i \times (n-i))$–matrix, and $\tilde{\mu} := (\mu_2, \ldots, \mu_{n-i})$.

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- *Moreover,* $S_{i,k}$ *is equidimensional of dimension* $D_i := i(n-i) - 1$ *and given as a transversal intersection of* $n+1$ *equations of degree* d *which have a circuit representation of size* $O(L + n + i(n-i))$

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- *In particular,* S_i *is a smooth subvariety of* $E_{i/\sim}$ *and the varieties* $S_{i,k}$, $1 \leq k \leq N_i$ *form an open atlas of* S_i.

The singular case: great miracle

The bipolar varieties

- The canonical projection

$$E_{i/\sim} \quad \longrightarrow \mathbb{A}^n$$

maps S_i surjectively onto the open set of smoothness $\mathcal{V} \setminus Sing\mathcal{V}$ of smooth points (recall: $\mathcal{V} := \{x \in \mathbb{A}^n \,|f(x) = 0\}$).

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- We can apply the technology developped previously for this case and consider of course its polar varieties (*generalized*). If sufficiently generic they form the bipolar varieties of de $\mathcal{V}$.
- The fibers $S_i(A)$ are (closed), generically smooth subvarieties dominating surjectively the (open) polar variety $\Delta_i\,(A)$.

The bipolar lattice

- The bipolar varieties are organized by decreasing codimension in strictly ascending dimension:

$$B_{i,D_i} \subset \cdots \subset B_{i,j} \subset \cdots B_{i,1} \subset B_{i,0} = S_i$$

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- If we let run also the index i from $n-1$ to 1, we obtain a tow–dimensional lattice of bipolar varieties
- A walk in this lattice is a path of length at most n, starting from a zero–dimensional bipolar variety $B_{i_1,D_{i_1}}$ to end up at an orbit variety $B_{i_2,0} = S_{i_2}$
- At each step either the index i or the codimension j decreases
- The bipolar varieties found under way along the walk, modulo suitable sections and indentifications, form an ascending sequence along which the dimension increases exactly by one. It is important to observe that their real traces are non empty, hence dense.

Algorithmic strategies

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- There exists a particular walk boiling down to the previously known algorithms treating the smooth case.

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Let $f(x_1, \ldots, x_n)$ *a polynomial of degree* $d \geq 2$ *defining as above complex and real hypersurfaces* V *and* $V_{\mathbb{R}}$*. Suppose that* f *is given by a straight–line program of size* L*.*

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Its sequential complexity is linear in L *and polynomial in* d*,* n *and an appropriate geometric quantity* $\delta_{\mathcal{W}}$*.*
This quantity is the maximal degree of the (complex) bipolar varieties of V *found along the walk. It is an intrinsic invariant of* V *and* $\mathcal{W}$*, which bounds also the number and the degree of the representative points exhibited by* $\mathcal{R}_{\mathcal{W}}$*.*

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If we choose $i := n-1$ we obtain an intrinsic variant of the so called "critical point" method, which is often used in a geometrically unstructured way with extrinsic complexity bounds.

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- Following the works by Castro–G–Heintz–Matera–Pardo, $d^{\Omega(n)}$ is also a lower bound for the worst case complexity of elimination problems like the one under consideration
- The only way out of this dilemma is the introduction of intrinsic complexity measures like $\delta_{\mathcal{W}}$.

Conclusion:

THE SAME COMPLEXITY CLASS!

KRONECKER vs. GRÖBNER using MAGMA 2.13
Polynomial equations from the design of wavelet filters
PhD Thesis of Lutz Lehmann, advisor Bernd Bank

δ	δ^*	ktime	kmem	gtime	gmem
12	6	$1.5\,s$	$3\,MB$	$1.2\,s$	$1\,MB$
12	8	$7\,s$	$3\,MB$	$0.3\,s$	$6\,MB$
54	22	$1\,m\;20\,s$	$7\,MB$	$3\,m\;10s$	$38\,MB$
28	10	$16\,s$	$9\,MB$	$6.5s$	$9\,MB$
28	10	$60\,s$	$10\,MB$	$15\,s$	$10\,MB$
136	24	$30\,m$	$50\,MB$	$> 5\,h$	$> 800\,MB$
136	26	$1h\;5\,m$	$75\,MB$	$> 5\,h$	$> 300\,MB$
32	6	$17\,s$	$7\,MB$	$2\,m\;30\,s$	$17\,MB$
32	10	$45\,s$	$7\,MB$	$67\,s$	$21\,MB$
168	36	$1h\;40\,m$	$98\,MB$	$> 5\,h$	$> 300\,MB$

δ: number of complex solutions k ... : Kronecker (Lecerf, Lehmann)
δ^*: number of real solutions g ... : Gröbner (Steel, F4 Faugère)

THANK YOU FOR YOUR ATTENTION!