

# Complexity of Bézout's Theorem and the Condition Number

Jean-Pierre Dedieu

Institut de Mathématiques de Toulouse, France

Fields Institute, Thematic Program on the Foundations of  
Computational Mathematics, July-December 2009

# Summary

- 1 Geometry for the polynomial system solving problem
- 2 Condition numbers
- 3 Distribution of the condition number
- 4 Complexity on the average
- 5 The approach based on the condition metric
- 6 Self-convexity

# Motivations

Our aim is to establish general principles for solving efficiently systems of polynomial equations. Very roughly speaking, algorithms for solving such systems can be divided into two classes: one algebraic with methods based on symbolic computation, the other one based on numerical analysis and approximation. In these lectures our interest focuses on this numerical approach.

In the first lecture we describe the general principles of continuation methods, and their numerical counterparts: the predictor-corrector algorithms. We estimate the complexity of such methods via the condition number analysis.

The second lecture is devoted to the probabilistic aspects of such questions. What can be said for random polynomial systems ? What is the distribution of the condition number ? What is the complexity of path-following methods on the average ?

In the third lecture we emphasize on a new approach, based on the condition metric (to be defined), and we list some open problems.

In the first lecture we describe the general principles of continuation methods, and their numerical counterparts: the predictor-corrector algorithms. We estimate the complexity of such methods via the condition number analysis.

The second lecture is devoted to the probabilistic aspects of such questions. What can be said for random polynomial systems ? What is the distribution of the condition number ? What is the complexity of path-following methods on the average ?

In the third lecture we emphasize on a new approach, based on the condition metric (to be defined), and we list some open problems.

In the first lecture we describe the general principles of continuation methods, and their numerical counterparts: the predictor-corrector algorithms. We estimate the complexity of such methods via the condition number analysis.

The second lecture is devoted to the probabilistic aspects of such questions. What can be said for random polynomial systems ? What is the distribution of the condition number ? What is the complexity of path-following methods on the average ?

In the third lecture we emphasize on a new approach, based on the condition metric (to be defined), and we list some open problems.

# The space of homogeneous polynomial system

Let  $\mathcal{H}_d$  be the vector space of **homogeneous polynomials**

$$f : \mathbb{C}^{n+1} \rightarrow \mathbb{C},$$

$f = (f_1, \dots, f_n)$  in the variable  $z = (z_0, z_1, \dots, z_n)$ , with degree  $d = (d_1, \dots, d_n)$ , so that  $f_i$  has degree  $d_i$ , and

$$f_i(z_0, z_1, \dots, z_n) = \sum_{\alpha_0 + \alpha_1 + \dots + \alpha_n = d_i} a_{i,\alpha} z_0^{\alpha_0} z_1^{\alpha_1} \dots z_n^{\alpha_n}$$

or, in the simpler form,

$$f_i(z) = \sum_{|\alpha|=d_i} a_{i,\alpha} z^\alpha.$$

The space  $\mathcal{H}_d$  is equipped with the Hermitian inner product

$$\langle f, g \rangle_d = \sum_{i=1}^n \langle f_i, g_i \rangle_{d_i} = \sum_{i=1}^n \sum_{|\alpha|=d_i} \frac{\alpha_0! \alpha_1! \dots \alpha_n!}{d_i!} a_{i,\alpha} \overline{b_{i,\alpha}}$$

with  $g_i(z) = \sum_{|\alpha|=d_i} b_{i,\alpha} z^\alpha$ .

This inner product is **unitarily invariant**:

$$\langle f \circ u, g \circ u \rangle_d = \langle f, g \rangle_d$$

for any unitary transformation  $u : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ .



# The solution variety

The **solution variety** for the polynomial solving problem is

$$\mathcal{V}_d = \{(f, \zeta) \in \mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C}) : f(\zeta) = 0\}.$$

Here  $\mathbb{P}(\mathcal{H}_d)$  denotes the projective space associated with  $\mathcal{H}_d$  and  $\mathbb{P}_n(\mathbb{C})$  the projective space associated with  $\mathbb{C}^{n+1}$ .

We take projective spaces instead of vector spaces because, by homogeneity, the zeros of a given  $f$  are lines through the origin in  $\mathbb{C}^{n+1}$  that is points in  $\mathbb{P}_n(\mathbb{C})$ , and the zeros of  $f$  and a multiple  $\lambda f$  are the same.

The solution variety is a **submanifold** in  $\mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C})$  and

$$\dim \mathcal{V}_d = \dim \mathbb{P}(\mathcal{H}_d).$$

# The solution variety

The **solution variety** for the polynomial solving problem is

$$\mathcal{V}_d = \{(f, \zeta) \in \mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C}) : f(\zeta) = 0\}.$$

Here  $\mathbb{P}(\mathcal{H}_d)$  denotes the projective space associated with  $\mathcal{H}_d$  and  $\mathbb{P}_n(\mathbb{C})$  the projective space associated with  $\mathbb{C}^{n+1}$ .

We take projective spaces instead of vector spaces because, by homogeneity, the zeros of a given  $f$  are lines through the origin in  $\mathbb{C}^{n+1}$  that is points in  $\mathbb{P}_n(\mathbb{C})$ , and the zeros of  $f$  and a multiple  $\lambda f$  are the same.

The solution variety is a **submanifold** in  $\mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C})$  and

$$\dim \mathcal{V}_d = \dim \mathbb{P}(\mathcal{H}_d).$$

# The solution variety

The **solution variety** for the polynomial solving problem is

$$\mathcal{V}_d = \{(f, \zeta) \in \mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C}) : f(\zeta) = 0\}.$$

Here  $\mathbb{P}(\mathcal{H}_d)$  denotes the projective space associated with  $\mathcal{H}_d$  and  $\mathbb{P}_n(\mathbb{C})$  the projective space associated with  $\mathbb{C}^{n+1}$ .

We take projective spaces instead of vector spaces because, by homogeneity, the zeros of a given  $f$  are lines through the origin in  $\mathbb{C}^{n+1}$  that is points in  $\mathbb{P}_n(\mathbb{C})$ , and the zeros of  $f$  and a multiple  $\lambda f$  are the same.

The solution variety is a **submanifold** in  $\mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C})$  and

$$\dim \mathcal{V}_d = \dim \mathbb{P}(\mathcal{H}_d).$$

# Well-posed and ill-posed problems

A problem  $(f, \zeta) \in \mathcal{V}_d$  is **well-posed** when  $\zeta$  is a simple root of the system  $f$  or, equivalently, when

$$\text{rank } Df(\zeta) = n$$

or, equivalently, when the derivative of the first projection

$$\Pi_1 : \mathcal{V}_d \rightarrow \mathbb{P}(\mathcal{H}_d), \quad \Pi_1(f, \zeta) = f$$

$$D\Pi_1(f, \zeta) : T_{(f, \zeta)}\mathcal{V}_d \rightarrow T_f\mathbb{P}(\mathcal{H}_d)$$

is an isomorphism.

# Well-posed and ill-posed problems

A problem  $(f, \zeta) \in \mathcal{V}_d$  is **well-posed** when  $\zeta$  is a simple root of the system  $f$  or, equivalently, when

$$\text{rank } Df(\zeta) = n$$

or, equivalently, when the derivative of the first projection

$$\Pi_1 : \mathcal{V}_d \rightarrow \mathbb{P}(\mathcal{H}_d), \quad \Pi_1(f, \zeta) = f$$

$$D\Pi_1(f, \zeta) : T_{(f, \zeta)}\mathcal{V}_d \rightarrow T_f\mathbb{P}(\mathcal{H}_d)$$

is an isomorphism.

Otherwise,  $(f, \zeta)$  is said **ill-posed**. The set of ill-posed problems is the **critical variety** denoted by

$$\Sigma' = \{(f, \zeta) \in \mathcal{V}_d : \text{rank } Df(\zeta) < n\}.$$

Its projection onto the space of systems is the **discriminant variety**

$$\Sigma = \Pi_1(\Sigma') = \{f \in \mathbb{P}(\mathcal{H}_d) : \exists \zeta \text{ with } (f, \zeta) \in \Sigma'\}.$$

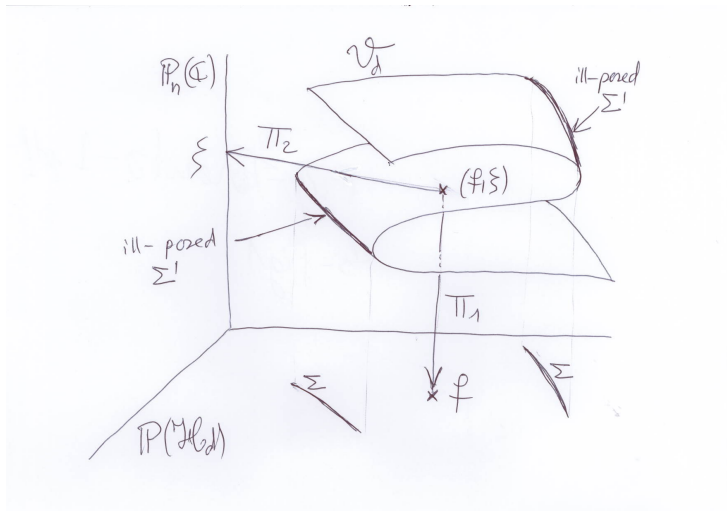
Otherwise,  $(f, \zeta)$  is said **ill-posed**. The set of ill-posed problems is the **critical variety** denoted by

$$\Sigma' = \{(f, \zeta) \in \mathcal{V}_d : \text{rank } Df(\zeta) < n\}.$$

Its projection onto the space of systems is the **discriminant variety**

$$\Sigma = \Pi_1(\Sigma') = \{f \in \mathbb{P}(\mathcal{H}_d) : \exists \zeta \text{ with } (f, \zeta) \in \Sigma'\}.$$

# The solution variety





# The continuation method

Let  $(f_0, \zeta_0) \in \mathcal{V}_d \setminus \Sigma'$  be given. For example

$$f_{0,i}(z) = z_0^{d_i-1} z_i, \quad \zeta_0 = e_0 = (1, 0, \dots, 0).$$

Consider a smooth path

$$f_t \in \mathbb{P}(\mathcal{H}_d) \setminus \Sigma, \quad 0 \leq t \leq 1,$$

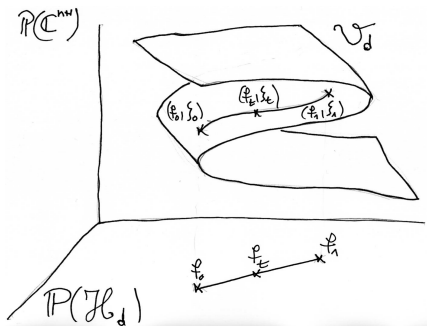
connecting  $f_0$  to a target system  $f_1$ . For example the **affine homotopy**

$$f_t = (1 - t)f_0 + tf_1.$$

Our objective is to find a zero  $\zeta_1$  of  $f_1$ .

According to the implicit function theorem, we can lift the path  $f_t \in \mathbb{P}(\mathcal{H}_d) \setminus \Sigma$  into a unique path

$$(f_t, \zeta_t) \in \mathcal{V}_d \setminus \Sigma'.$$



# Predictor-Corrector Methods 1

In practice this program is realized via a **discretized problem** and an approximation method:

The interval  $[0, 1]$  is discretized in

$$0 = t_0 < t_1 < \dots < t_k = 1$$

and the path  $(f_t, \zeta_t) \in \mathcal{V}_d \setminus \Sigma'$  by a sequence of pairs

$$(f_{t_i}, z_{t_i}) \in \mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C}), \quad 0 \leq i \leq k,$$

where  $z_{t_0} = \zeta_0$  and  $z_{t_{i+1}}$  is computed from  $z_{t_i}$  by a **predictor-corrector method**.

# Predictor-Corrector Methods 1

In practice this program is realized via a **discretized problem** and an approximation method:

The interval  $[0, 1]$  is discretized in

$$0 = t_0 < t_1 < \dots < t_k = 1$$

and the path  $(f_t, \zeta_t) \in \mathcal{V}_d \setminus \Sigma'$  by a sequence of pairs

$$(f_{t_i}, z_{t_i}) \in \mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C}), \quad 0 \leq i \leq k,$$

where  $z_{t_0} = \zeta_0$  and  $z_{t_{i+1}}$  is computed from  $z_{t_i}$  by a **predictor-corrector method**.

# Predictor-Corrector Methods 1

In practice this program is realized via a **discretized problem** and an approximation method:

The interval  $[0, 1]$  is discretized in

$$0 = t_0 < t_1 < \dots < t_k = 1$$

and the path  $(f_t, \zeta_t) \in \mathcal{V}_d \setminus \Sigma'$  by a sequence of pairs

$$(f_{t_i}, z_{t_i}) \in \mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C}), \quad 0 \leq i \leq k,$$

where  $z_{t_0} = \zeta_0$  and  $z_{t_{i+1}}$  is computed from  $z_{t_i}$  by a **predictor-corrector method**.

# Predictor-Corrector Methods 1

In practice this program is realized via a **discretized problem** and an approximation method:

The interval  $[0, 1]$  is discretized in

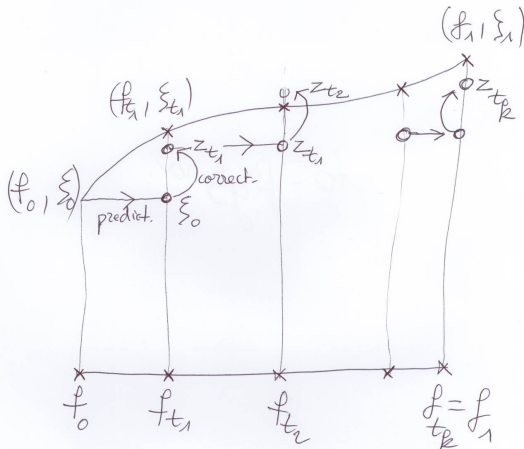
$$0 = t_0 < t_1 < \dots < t_k = 1$$

and the path  $(f_t, \zeta_t) \in \mathcal{V}_d \setminus \Sigma'$  by a sequence of pairs

$$(f_{t_i}, z_{t_i}) \in \mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C}), \quad 0 \leq i \leq k,$$

where  $z_{t_0} = \zeta_0$  and  $z_{t_{i+1}}$  is computed from  $z_{t_i}$  by a **predictor-corrector method**.

# Predictor-Corrector Methods bis



# Predictor-Corrector Methods

When the points in the mesh  $(t_i)$  are close enough, the corresponding pairs  $(f_{t_i}, z_{t_i})$  stay close to the path  $(f_t, \zeta_t) \in \mathcal{V}_d \setminus \Sigma'$  and  $z_{t_i}$  is a good approximation of  $\zeta_{t_i}$ .

We measure the **complexity** of this process as the number  $k$  of points in the mesh.



# Predictor-Corrector Methods

When the points in the mesh  $(t_i)$  are close enough, the corresponding pairs  $(f_{t_i}, z_{t_i})$  stay close to the path  $(f_t, \zeta_t) \in \mathcal{V}_d \setminus \Sigma'$  and  $z_{t_i}$  is a good approximation of  $\zeta_{t_i}$ .

We measure the **complexity** of this process as the number  $k$  of points in the mesh.

# Newton's operator

The correction step in our predictor-corrector algorithm plays a crucial role! We define it by

$$z_{t_{i+1}} = N_{f_{t_{i+1}}}(z_{t_i})$$

where  $N_f$  is the **projective Newton operator** associated with  $f$ . This operator is defined by

$$N_f(z) = z - (Df(z) \mid_{T_z \mathbb{P}_n(\mathbb{C})})^{-1} f(z)$$

where  $T_z \mathbb{P}_n(\mathbb{C})$ , the tangent space at  $z \in \mathbb{P}_n(\mathbb{C})$ , is identified with

$$z^\perp = \{u \in \mathbb{C}^{n+1} : \langle u, z \rangle = 0\}.$$

# Newton's operator

The correction step in our predictor-corrector algorithm plays a crucial role! We define it by

$$z_{t_{i+1}} = N_{f_{t_{i+1}}}(z_{t_i})$$

where  $N_f$  is the **projective Newton operator** associated with  $f$ . This operator is defined by

$$N_f(z) = z - (Df(z) \mid_{T_z \mathbb{P}_n(\mathbb{C})})^{-1} f(z)$$

where  $T_z \mathbb{P}_n(\mathbb{C})$ , the tangent space at  $z \in \mathbb{P}_n(\mathbb{C})$ , is identified with

$$z^\perp = \{u \in \mathbb{C}^{n+1} : \langle u, z \rangle = 0\}.$$

# Newton's sequence

Since  $N_f(\lambda x) = \lambda N_f(x)$ , this operator is well-defined in  $\mathbb{P}_n(\mathbb{C})$ .

We say that  $x_0 \in \mathbb{P}_n(\mathbb{C})$  is an **approximate zero of**  $f \in \mathbb{P}(\mathcal{H}_d)$  **with actual zero**  $\zeta \in \mathbb{P}_n(\mathbb{C})$  when the Newton sequence defined by

$$x_{i+1} = N_f(x_i), \quad i \geq 0,$$

converges immediatly quadratically to  $\zeta$  that is when

$$d_P(\zeta, x_i) \leq \left(\frac{1}{2}\right)^{2^i - 1} d_P(\zeta, x_0).$$

Here  $d_P(x, y)$  is the distance in  $\mathbb{P}_n(\mathbb{C})$  defined by the sine of the angle made by the lines through  $x$  and  $y$ .

# Newton's sequence

Since  $N_f(\lambda x) = \lambda N_f(x)$ , this operator is well-defined in  $\mathbb{P}_n(\mathbb{C})$ .

We say that  $x_0 \in \mathbb{P}_n(\mathbb{C})$  is an **approximate zero of**  $f \in \mathbb{P}(\mathcal{H}_d)$  **with actual zero**  $\zeta \in \mathbb{P}_n(\mathbb{C})$  when the Newton sequence defined by

$$x_{i+1} = N_f(x_i), \quad i \geq 0,$$

converges immediatly quadratically to  $\zeta$  that is when

$$d_P(\zeta, x_i) \leq \left(\frac{1}{2}\right)^{2^i - 1} d_P(\zeta, x_0).$$

Here  $d_P(x, y)$  is the distance in  $\mathbb{P}_n(\mathbb{C})$  defined by the sine of the angle made by the lines through  $x$  and  $y$ .

# Gamma Theorem

A quantitative criterion to decide whether a given  $x$  is an approximate zero with actual zero  $\zeta$  is given by

$$d_P(\zeta, x) \leq \frac{3 - \sqrt{7}}{2\gamma(f, \zeta)}$$

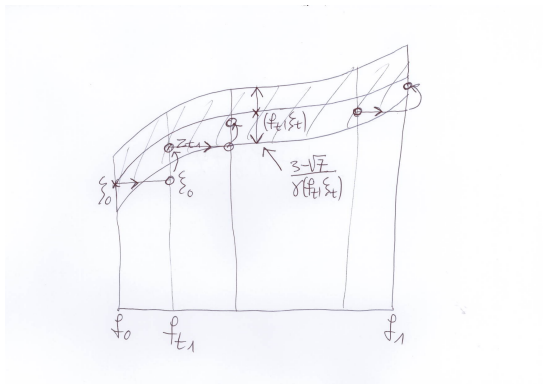
with

$$\gamma(f, \zeta) = \|\zeta\| \sup_{k \geq 2} \left\| \frac{(Df(\zeta)|_{\zeta^\perp})^{-1} D^k f(\zeta)}{k!} \right\|^{\frac{1}{k-1}}.$$

We now understand much better what we have to do:

# Tubular neighborhood

We want our predictor-corrector sequence  $(f_{t_i}, z_{t_i})$  to stay inside the tubular neighborhood of the path  $(f_t, \zeta_t)$  with size  $(3 - \sqrt{7})/(2\gamma(f_t, \zeta_t))$  so that each  $z_{t_i}$  is an approximate zero of  $f_{t_i}$  with actual zero  $\zeta_{t_i}$ .



# Definition of the condition number

The **condition number** measures the size of the first order variations of the solution of a problem in terms of the size of the first order variations of the problem. Let

$$\mathcal{S} : \text{Input Space} \rightarrow \text{Output Space}$$

be the (locally defined) solution map and let  $p$  be an instance of our problem. Then, we define the condition number at  $p$  as **the operator norm of the derivative of the solution map** at  $p$ .



# The case of homogeneous polynomial systems

For a well-posed problem  $(f, \zeta) \in \mathcal{V}_d \setminus \Sigma'$  the condition number is given by

$$\mu(f, \zeta) = \|f\| \left\| (Df(\zeta) |_{\zeta^\perp})^{-1} \operatorname{diag} \left( \|\zeta\|^{d_i-1} \right) \right\|$$

when the space  $\mathcal{H}_d$  is equipped with the Hermitian structure defined previously.

We will see in the second lecture that the condition number  $\mu(f, \zeta)$  is related to the distance from ill-posed problems like it is the case for matrices.

# The case of homogeneous polynomial systems

For a well-posed problem  $(f, \zeta) \in \mathcal{V}_d \setminus \Sigma'$  the condition number is given by

$$\mu(f, \zeta) = \|f\| \left\| (Df(\zeta)|_{\zeta^\perp})^{-1} \operatorname{diag} \left( \|\zeta\|^{d_i-1} \right) \right\|$$

when the space  $\mathcal{H}_d$  is equipped with the Hermitian structure defined previously.

We will see in the second lecture that the condition number  $\mu(f, \zeta)$  is related to the distance from ill-posed problems like it is the case for matrices.

# Gamma and the condition number

The last step before reaching our objective is given by the following estimate (Shub-Smale)

$$\gamma(f, \zeta) \leq \frac{D^2}{2} \mu(f, \zeta)$$

with  $D = \max d_i$ , called the **Higher Derivative Estimate**. Thus, the size of the tubular neighborhood of our path  $(f_t, \zeta_t)$  depends on the conditionning of the encountered problems.

Ill-conditioned instances implies a very small neighborhood and increases the complexity  $k$  of our algorithm. More precisely:

# Complexity of continuation methods

*(Shub-Smale) Given a path  $f_t \in \mathbb{P}(\mathcal{H}_d) \setminus \Sigma$ ,  $0 \leq t \leq 1$ , a zero  $\zeta_0$  of  $f_0$ , and the corresponding lifted path  $F(t) = (f_t, \zeta_t) \in \mathcal{V}_d \setminus \Sigma'$ , there exists a mesh  $0 = t_0 < t_1 < \dots < t_k = 1$  such that each  $z_{t_i}$  constructed by the Newton predictor-corrector algorithm is an approximate zero of  $f_{t_i}$  with actual zero  $\zeta_{t_i}$  with*

$$k \leq CD^2 \mu(F)^2 L_f.$$

*$C$  is an absolute constant,  $L_f$  is the length of the path  $f_t$  in  $\mathbb{P}(\mathcal{H}_d)$ , and*

$$\mu(F) = \sup_{0 \leq t \leq 1} \mu(f_t, \zeta_t)$$

*is the condition number of the path.*

# The sparse case

A polynomial system is **sparse** when it has few non-zero coefficients (**fewnomials**) or when its coefficients depend on a small number of parameters, for example

$$(x + a)^m + b)^n.$$

Let  $\mathcal{K}$  denote a set of such systems. We suppose it is a submanifold of small dimension in  $\mathbb{P}(\mathcal{H}_d)$ .

The **sparse solution variety** is

$$\mathcal{V}_{\mathcal{K}} = \{(f, \zeta) \in \mathcal{K} \times \mathbb{P}_n(\mathbb{C}) : f(\zeta) = 0\}.$$

We define the **sparse condition number**  $\mu_{\mathcal{K}}$  in only considering perturbations of the system which respect the sparse structure.

# The sparse case

A polynomial system is **sparse** when it has few non-zero coefficients (**fewnomials**) or when its coefficients depend on a small number of parameters, for example

$$(x + a)^m + b)^n.$$

Let  $\mathcal{K}$  denote a set of such systems. We suppose it is a submanifold of small dimension in  $\mathbb{P}(\mathcal{H}_d)$ .

The **sparse solution variety** is

$$\mathcal{V}_{\mathcal{K}} = \{(f, \zeta) \in \mathcal{K} \times \mathbb{P}_n(\mathbb{C}) : f(\zeta) = 0\}.$$

We define the **sparse condition number**  $\mu_{\mathcal{K}}$  in only considering perturbations of the system which respect the sparse structure.

# The sparse case bis

$$\mu(f, \zeta) = \sup \frac{\|DSol(f, \zeta)(\dot{f})\|}{\|\dot{f}\|}, \quad \dot{f} \in T_f \mathbb{P}(\mathcal{H}_d),$$

while

$$\mu_{\mathcal{K}}(f, \zeta) = \sup \frac{\|DSol(f, \zeta)(\dot{f})\|}{\|\dot{f}\|}, \quad \dot{f} \in T_f \mathcal{K}.$$

The second supremum is smaller than the first.

# The sparse case ter

Following the previous lines it may be shown that the complexity of continuation methods for a path

$$F(t) = (f_t, \zeta_t) \in \mathcal{V}_K, \quad 0 \leq t \leq 1,$$

is bounded by

$$CD^2_{\mu_K}(F)\mu(F)L_f.$$

Here  $C$  is a constant, and  $\mu_K(F) \leq \mu(F)$  the sparse and non-sparse condition numbers.



# References

The matter of this lecture is taken from

- ① M. SHUB, AND S. SMALE, *Complexity of Bézout's Theorem I: Geometric Aspects*, J. Am. Math. Soc. (1993) 6 pp. 459-501.
- ② J.-P. DEDIEU, *Condition number analysis for sparse polynomial systems*, In : Foundations of Computational Mathematics, F. Cucker, M. Shub editors, Springer 1997, 75-101.
- ③ L. BLUM, F. CUCKER, M. SHUB, AND S. SMALE, *Complexity and Real Computation*, Springer, 1998.

# Second Talk : Motivation

We have seen in the previous talk the important role played by the condition number in the complexity of predictor-corrector methods based on Newton method. Let us recall this bound:

$$CD^2 \max_{0 \leq t \leq 1} \mu(f_t, \zeta_t)^2 L_f.$$

In order to obtain estimates on the average for this complexity bound we have to study the distribution of the condition number.

# Second Talk : Motivation

We have seen in the previous talk the important role played by the condition number in the complexity of predictor-corrector methods based on Newton method. Let us recall this bound:

$$CD^2 \max_{0 \leq t \leq 1} \mu(f_t, \zeta_t)^2 L_f.$$

In order to obtain estimates on the average for this complexity bound we have to study the distribution of the condition number.

# Second Talk : Motivation

We have seen in the previous talk the important role played by the condition number in the complexity of predictor-corrector methods based on Newton method. Let us recall this bound:

$$CD^2 \max_{0 \leq t \leq 1} \mu(f_t, \zeta_t)^2 L_f.$$

In order to obtain estimates on the average for this complexity bound we have to study the distribution of the condition number.

# The Normalized Condition Number

For a well-posed problem  $(f, \zeta) \in \mathcal{V}_d \setminus \Sigma'$  the condition number is equal to

$$\mu(f, \zeta) = \|f\| \left\| (Df(\zeta) |_{\zeta^\perp})^{-1} \operatorname{diag} \left( \|\zeta\|^{d_i-1} \right) \right\|.$$

We define the **normalized condition number** by

$$\mu_{\text{norm}}(f, \zeta) = \|f\| \left\| (Df(\zeta) |_{\zeta^\perp})^{-1} \operatorname{diag} \left( d_i^{1/2} \|\zeta\|^{d_i-1} \right) \right\|.$$

We have

$$\mu(f, \zeta) \leq \mu_{\text{norm}}(f, \zeta) \leq D^{1/2} \mu(f, \zeta).$$

# The Normalized Condition Number

For a well-posed problem  $(f, \zeta) \in \mathcal{V}_d \setminus \Sigma'$  the condition number is equal to

$$\mu(f, \zeta) = \|f\| \left\| (Df(\zeta) |_{\zeta^\perp})^{-1} \operatorname{diag} \left( \|\zeta\|^{d_i-1} \right) \right\|.$$

We define the **normalized condition number** by

$$\mu_{\text{norm}}(f, \zeta) = \|f\| \left\| (Df(\zeta) |_{\zeta^\perp})^{-1} \operatorname{diag} \left( d_i^{1/2} \|\zeta\|^{d_i-1} \right) \right\|.$$

We have

$$\mu(f, \zeta) \leq \mu_{\text{norm}}(f, \zeta) \leq D^{1/2} \mu(f, \zeta).$$

# The Normalized Condition Number bis

We have defined the condition number  $\mu(f, \zeta)$  as the **operator norm** of the derivative of the solution map at  $(f, \zeta) \in V$ .

The normalized condition number obey to the same law: but we have to change the metric structure in  $\mathbb{P}(\mathcal{H}_d)$  taking

$$\langle f, g \rangle_{norm} = \sum_{i=1}^n d_i^{-1} \langle f_i, g_i \rangle_i$$

instead of

$$\langle f, g \rangle_{norm} = \sum_{i=1}^n \langle f_i, g_i \rangle_i.$$

# The Condition Number: matrices

Consider the solution variety associated to the problem: solve the linear system

$$Ax = b.$$

Here  $A$  is given,  $b$  the problem instance,  $x = A^{-1}b$  the solution and

$$b \in \mathbb{C}^n \rightarrow A^{-1}b \in \mathbb{C}^n$$

the solution map. We equip  $\mathbb{C}^n$  of the usual Hermitian structure so that the associated condition number is

$$\mu(b, x) = \|A^{-1}\|_2.$$



# The Condition Number Theorem: matrices

Eckart-Young Theorem relate this condition number to the Frobenius distance of  $A$  to singular matrices:

$$\|A^{-1}\|_2 = \frac{1}{d_F(A, \Sigma)},$$

$$\Sigma = \{A \in \mathbb{C}^{n \times n}\}$$

# The Condition Number Theorem

The main interest of the normalized condition number is given by the **Condition Number Theorem** which relates  $\mu_{\text{norm}}(f, \zeta)$  and the distance of  $(f, \zeta)$  to a certain set of ill-posed problems. Define

$$\rho(f, \zeta) = d_P((f, \zeta), \Sigma_\zeta)$$

where

$$\Sigma_\zeta = \Sigma' \cap (\mathbb{P}(\mathcal{H}_d) \times \{\zeta\}) = \Sigma' \cap \Pi_2^{-1}(\zeta)$$

is the set of ill-posed problems with a multiple root at  $\zeta$ , and

$$d_P((f, \zeta), \Sigma_\zeta) = \min_{(g, \zeta) \in \Sigma_\zeta} d_P(f, g)$$

as previously.

*(Shub-Smale, Condition Number Theorem)*

$$\mu_{\text{norm}}(\mathbf{f}, \zeta) = \frac{1}{\rho(\mathbf{f}, \zeta)}.$$

# Probability measures

The **probability measures** on  $\mathbb{P}_n(\mathbb{C})$ , and on  $\mathbb{P}(\mathcal{H}_d)$  are induced by the underlying Riemannian structures given by the Hermitian structures on  $\mathbb{C}^{n+1}$  and  $\mathcal{H}_d$ . These Riemannian structures give a product structure on  $\mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C})$  and an induced structure on  $\mathcal{V}_d$ .

They are invariant under the action of the unitary group  $\mathbb{U}_{n+1}$

$$((f, \zeta), u) \in \mathcal{V}_d \times \mathbb{U}_{n+1} \rightarrow (f \circ u, u^*(\zeta)) \in \mathcal{V}_d.$$

# Probability measures

The **probability measures** on  $\mathbb{P}_n(\mathbb{C})$ , and on  $\mathbb{P}(\mathcal{H}_d)$  are induced by the underlying Riemannian structures given by the Hermitian structures on  $\mathbb{C}^{n+1}$  and  $\mathcal{H}_d$ . These Riemannian structures give a product structure on  $\mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C})$  and an induced structure on  $\mathcal{V}_d$ .

They are invariant under the action of the unitary group  $\mathbb{U}_{n+1}$

$$((f, \zeta), u) \in \mathcal{V}_d \times \mathbb{U}_{n+1} \rightarrow (f \circ u, u^*(\zeta)) \in \mathcal{V}_d.$$

# Distribution of the condition number in $\mathcal{V}_d$

(Shub-Smale) If  $n > 1$  and  $D \geq 1$ , for any  $\varepsilon > 0$

$$\text{Prob} \left\{ (f, \zeta) \in \mathcal{V}_d : \mu_{\text{norm}}(f, \zeta) \geq \frac{1}{\varepsilon} \right\} \leq \varepsilon^2 n N C$$

( $C$  is an absolute constant and  $N = \dim \mathcal{H}_d$ ).

# Distribution of the condition number in $\mathbb{P}(\mathcal{H}_d)$

Define

$$\mu_{\text{norm}}(f) = \max_{f(\zeta)=0} \mu_{\text{norm}}(f, \zeta).$$

*(Shub-Smale) When  $n > 1$  and  $D \geq 1$ , for any  $0 < \varepsilon < 1/\sqrt{n}$ , one has*

$$\text{Prob} \left\{ f \in \mathbb{P}(\mathcal{H}_d) : \mu_{\text{norm}}(f) \geq \frac{1}{\varepsilon} \right\} \leq \frac{\varepsilon^4 n^3 N^2 \mathcal{D}}{4}$$

*where  $\mathcal{D} = d_1 d_2 \dots d_n$  is the Bézout number.*

# Complexity on the average

Given  $(f_0, \zeta_0) \in \mathcal{V}_d \setminus \Sigma'$  we consider the path  $tf_1 + (1 - t)f_0$ ,  $0 \leq t \leq 1$  to reach an approximate zero of a given  $f_1$  and the corresponding Newton predictor-corrector method.

*(Shub-Smale) Given  $0 < \sigma < 1$ , there exists  $(f_0, \zeta_0) \in \mathcal{V}_d \setminus \Sigma'$  such that  $k$  steps are sufficient to find an approximate zero of any  $f_1 \in \mathbb{P}(\mathcal{H}_d)$  with probability of failure  $\sigma$  and*

$$k \leq \frac{CN^3}{\sigma^{1-\varepsilon}}, \quad \varepsilon = \frac{1}{\log \mathcal{D}}$$

*(or  $CN^4/\sigma^{1-\varepsilon}$  if some  $d_i = 1$  or  $n \leq 4$ ).*



# Complexity on the average

Given  $(f_0, \zeta_0) \in \mathcal{V}_d \setminus \Sigma'$  we consider the path  $tf_1 + (1 - t)f_0$ ,  $0 \leq t \leq 1$  to reach an approximate zero of a given  $f_1$  and the corresponding Newton predictor-corrector method.

*(Shub-Smale) Given  $0 < \sigma < 1$ , there exists  $(f_0, \zeta_0) \in \mathcal{V}_d \setminus \Sigma'$  such that  $k$  steps are sufficient to find an approximate zero of any  $f_1 \in \mathbb{P}(\mathcal{H}_d)$  with probability of failure  $\sigma$  and*

$$k \leq \frac{CN^3}{\sigma^{1-\varepsilon}}, \quad \varepsilon = \frac{1}{\log \mathcal{D}}$$

*(or  $CN^4/\sigma^{1-\varepsilon}$  if some  $d_i = 1$  or  $n \leq 4$ ).*

- A priori, the initial pair  $(f_0, \zeta_0)$  depends on the degree  $d = (d_1, \dots, d_n)$  and on the probability of failure  $\sigma$ .
- The proof given by the authors does not give an explicit construction for  $(f_0, \zeta_0)$ .
- They conjecture that

$$f_0 = (x_0^{d_i-1} x_i)_{1 \leq i \leq n}, \quad \zeta_0 = e_0 = (1, 0, \dots, 0)$$

is a good initial guess.

# Complexity on the average, bis

From this theorem we obtain **non uniform algorithm in probabilistic polynomial time** for Bézout's Theorem:

*(Shub-Smale) Fixing  $d$ , the average number to find an approximate zero of  $f \in \mathbb{P}(\mathcal{H}_d)$  is less than  $CN^4$  unless  $n \leq 4$  or some  $d_i = 1$ . In this latter case we get  $CN^5$ .*

# $\sigma$ -efficient pairs

Given  $\sigma > 0$ , a pair  $(f_0, z_0) \in \mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C})$  is said  **$\sigma$ -efficient** when there is a polynomial

$$k = k(\sigma^{-1}, n, N, d)$$

such that, for any  $f_1 \in \mathbb{P}(\mathcal{H}_d)$ ,  $k$  steps of Newton predictor-corrector method associated with the affine homotopy  $tf_1 + (1 - t)f_0$ , with initial pair  $(f_0, z_0)$ , are sufficient to find an approximate zero of  $f_1$  with probability of failure  $\sigma$ .

Shub-Smale's Theorem shows that  $\sigma$ -efficient pairs always exists.

# $\sigma$ -efficient pairs

Given  $\sigma > 0$ , a pair  $(f_0, z_0) \in \mathbb{P}(\mathcal{H}_d) \times \mathbb{P}_n(\mathbb{C})$  is said  **$\sigma$ -efficient** when there is a polynomial

$$k = k(\sigma^{-1}, n, N, d)$$

such that, for any  $f_1 \in \mathbb{P}(\mathcal{H}_d)$ ,  $k$  steps of Newton predictor-corrector method associated with the affine homotopy  $tf_1 + (1 - t)f_0$ , with initial pair  $(f_0, z_0)$ , are sufficient to find an approximate zero of  $f_1$  with probability of failure  $\sigma$ .

Shub-Smale's Theorem shows that  $\sigma$ -efficient pairs always exists.

Given a multi-degree  $d$ , Beltrán-Pardo give an explicit construction of a subset of the solution variety  $\mathcal{G}_d \subset \mathcal{V}_d$  that share the common zero  $e_0 = (1, 0, \dots, 0)$  with the following property:

*(Beltrán-Pardo) For any  $0 < \sigma < 1$ , the probability that a randomly chosen pair  $(f_0, e_0) \in \mathcal{G}_d$  is  $\sigma$ -efficient is greater than*

$$1 - \sigma.$$

*For these  $\sigma$ -efficient pairs  $(f_0, e_0) \in \mathcal{G}_d$ , and for any  $f_1 \in \mathbb{P}(\mathcal{H}_d)$ ,*

$$k = Cn^5 N^2 d^4 \sigma^{-2}$$

*steps of Newton predictor-corrector method associated with the affine homotopy  $tf_1 + (1 - t)f_0$ , and the initial pair  $(f_0, e_0)$ , are sufficient to find an approximate zero of  $f_1$  with probability of failure  $\sigma$ .*

Beltrán-Pardo construction gives a **uniform algorithm in probabilistic polynomial time** to find an approximate zero of a given polynomial system.



# References

- M. SHUB, AND S. SMALE, *Complexity of Bézout's Theorem I: Geometric Aspects*, J. Am. Math. Soc. (1993) 6 pp. 459-501.
- M. SHUB, AND S. SMALE, *Complexity of Bézout's Theorem II : Volumes and Probabilities*, in : F. Eyssette, A. Galligo Eds. Computational Algebraic Geometry, Progress in Mathematics. Vol. 109, Birkhäuser, (1993).
- M. SHUB, AND S. SMALE, *Complexity of Bézout's Theorem V : Polynomial Time*, Theoretical Computer Science, 133, 141-164 (1994).

- C. BELTRÁN, AND L. M. PARDO, *On Smale's 17th Problem: a Probabilistic Positive Solution*. FOCM, (2008) 1-43.
- C. BELTRÁN, AND L. M. PARDO, *Smale's 17th Problem: Average Polynomial Time to Compute Affine and Projective Solutions*. J. AMS, 22 (2009) 363-385.

# Third Talk : Motivation

The classical approach to continuation methods to solve polynomial systems may be summarized in the following steps

- Consider **in the space of systems** a path connecting an initial system with a target system,
- **Lift it in the solution variety**, starting with a given zero of the initial system to obtain a zero for the target system,
- Follow the lifted path using a **predictor-corrector method**.

# Motivation bis

The complexity of these methods is dominated by the condition number of the encountered problems along the **lifted path**, and, even if the initial and target systems are well-conditionned, this lifted path may be very close to the critical variety giving ill-conditionned problems, and slowing our algorithm.

To avoid this difficulty we search for a path in the solution variety **as short as possible** which stays **as far as possible from the critical variety** !

How to proceed ?

# Condition length

The length of a path  $F(t) = (f_t, \zeta_t)$ ,  $0 \leq t \leq 1$ , for the usual Riemannian structure in  $\mathcal{V}_d$  is given by

$$L(F) = \int_0^1 \left\| \frac{d}{dt} (f_t, \zeta_t) \right\|_{(f_t, \zeta_t)} dt$$

with

$$\left\| (\dot{f}, \dot{\zeta}) \right\|_{(f, \zeta)}^2 = \frac{\left\| \dot{f} \right\|^2}{\left\| f \right\|^2} + \frac{\left\| \dot{\zeta} \right\|^2}{\left\| \zeta \right\|^2}$$

for any  $(\dot{f}, \dot{\zeta}) \in T_{(f, \zeta)} \mathcal{V}_d$ .

# Condition length bis

The **length in the condition metric** also called **condition length** is defined by

$$L_{\kappa}(F) = \int_0^1 \left\| \frac{d}{dt} (f_t, \zeta_t) \right\|_{(f_t, \zeta_t)} \mu_{\text{norm}}(f_t, \zeta_t) dt.$$

The **condition distance** between two pairs  $F_0$ , and  $F_1 \in \mathcal{V}_d \setminus \Sigma'$  is given as the infimum of the lengths of paths with endpoints  $F_0$  and  $F_1$ :

$$d_{\kappa}(F_0, F_1) = \inf \{ L_{\kappa}(F) : F(0) = F_0, F(1) = F_1 \}.$$

A curve which realizes this infimum, parametrized by arc-length, is called a **minimizing condition geodesic**.

# Condition length bis

The **length in the condition metric** also called **condition length** is defined by

$$L_{\kappa}(F) = \int_0^1 \left\| \frac{d}{dt} (f_t, \zeta_t) \right\|_{(f_t, \zeta_t)} \mu_{\text{norm}}(f_t, \zeta_t) dt.$$

The **condition distance** between two pairs  $F_0$ , and  $F_1 \in \mathcal{V}_d \setminus \Sigma'$  is given as the infimum of the lengths of paths with endpoints  $F_0$  and  $F_1$ :

$$d_{\kappa}(F_0, F_1) = \inf \{ L_{\kappa}(F) : F(0) = F_0, F(1) = F_1 \}.$$

A curve which realizes this infimum, parametrized by arc-length, is called a **minimizing condition geodesic**.

# Condition length bis

The **length in the condition metric** also called **condition length** is defined by

$$L_{\kappa}(F) = \int_0^1 \left\| \frac{d}{dt} (f_t, \zeta_t) \right\|_{(f_t, \zeta_t)} \mu_{\text{norm}}(f_t, \zeta_t) dt.$$

The **condition distance** between two pairs  $F_0$ , and  $F_1 \in \mathcal{V}_d \setminus \Sigma'$  is given as the infimum of the lengths of paths with endpoints  $F_0$  and  $F_1$ :

$$d_{\kappa}(F_0, F_1) = \inf \{ L_{\kappa}(F) : F(0) = F_0, F(1) = F_1 \}.$$

A curve which realizes this infimum, parametrized by arc-length, is called a **minimizing condition geodesic**.



# Condition length ter

Condition length and condition geodesics are related to the **condition Riemannian structure** defined on  $\mathcal{V}_d \setminus \Sigma'$  by

$$\langle \cdot, \cdot \rangle_{\kappa, (f, \zeta)} = \langle \cdot, \cdot \rangle_{(f, \zeta)} \mu_{\text{norm}}(f, \zeta)^2.$$

Notice that  $\mu_{\text{norm}}(f, \zeta)$  is **not a smooth function** on  $\mathcal{V}_d \setminus \Sigma'$ ; however it is **locally Lipschitz**.

# The solution variety as a length space

$\mathcal{V}_d \setminus \Sigma'$  equipped with the condition metric  $d_\kappa$  is a complete metric space. Moreover, given two pairs  $F_0$  and  $F_1$  in this space, there exists an absolutely continuous minimizing condition geodesic  $F(t)$  such that

$$d_\kappa(F_0, F_1) = L_\kappa(F), \quad F(0) = F_0, \quad F(1) = F_1.$$

Such a condition geodesic is  $C^1$  with a Lipschitz derivative, that is  $W^{2,\infty}$ .

# Complexity in the condition length

The interest of considering minimizing condition geodesics appear clearly in the following:

*(Shub) Given a smooth path  $F = (f_t, \zeta_t) \in \mathcal{V}_d \setminus \Sigma'$ ,  $0 \leq t \leq 1$ ,*

$$k \leq CD^{3/2} L_\kappa(F)$$

*steps are sufficient to follow approximately this path.*

# Complexity in the condition length

The interest of considering minimizing condition geodesics appear clearly in the following:

*(Shub) Given a smooth path  $F = (f_t, \zeta_t) \in \mathcal{V}_d \setminus \Sigma'$ ,  $0 \leq t \leq 1$ ,*

$$k \leq CD^{3/2} L_\kappa(F)$$

*steps are sufficient to follow approximately this path.*

# Complexity in the condition length bis

Since

$$L_{\kappa}(F) \leq \mu_{\text{norm}}(F)L(F) \leq \mu_{\text{norm}}(F)\mu(F)L_f \leq D^{1/2}\mu(F)^2L_f$$

the bound in terms of the condition length is an improvement of  
Bézout 1 Main Theorem:

$$CD^{3/2}L_{\kappa}(F) \leq CD^2\mu(F)^2L_f.$$

To improve complexity of path-following methods, this bound suggests to follow the minimizing condition geodesics in the solution variety !

# Short paths

The following shows that there exist very short paths connecting two given pairs in  $\mathcal{V}_d \setminus \Sigma'$ : Let us define

$$g = (d_1^{1/2} z_0^{d_1-1} z_1, \dots, d_n^{1/2} z_0^{d_n-1} z_n), \text{ and } e_0 = (1, 0, \dots, 0).$$

*(Beltrán-Shub) Given  $(f, \zeta) \in \mathcal{V}_d \setminus \Sigma'$ , there exists a path in the condition metric with endpoints  $(g, e_0)$  and  $(f, \zeta)$  with condition length*

$$L_K \leq 9nD^{3/2} + 2\sqrt{n} \ln \left( \frac{\mu_{\text{norm}}(f, \zeta)}{\sqrt{n}} \right).$$

Following such a path will give a complexity in the logarithm of the condition number instead of its square !

# Short paths

The following shows that there exist very short paths connecting two given pairs in  $\mathcal{V}_d \setminus \Sigma'$ : Let us define

$$g = (d_1^{1/2} z_0^{d_1-1} z_1, \dots, d_n^{1/2} z_0^{d_n-1} z_n), \text{ and } e_0 = (1, 0, \dots, 0).$$

*(Beltrán-Shub) Given  $(f, \zeta) \in \mathcal{V}_d \setminus \Sigma'$ , there exists a path in the condition metric with endpoints  $(g, e_0)$  and  $(f, \zeta)$  with condition length*

$$L_K \leq 9nD^{3/2} + 2\sqrt{n} \ln \left( \frac{\mu_{\text{norm}}(f, \zeta)}{\sqrt{n}} \right).$$

Following such a path will give a complexity in the logarithm of the condition number instead of its square !

# Short paths

The following shows that there exist very short paths connecting two given pairs in  $\mathcal{V}_d \setminus \Sigma'$ : Let us define

$$g = (d_1^{1/2} z_0^{d_1-1} z_1, \dots, d_n^{1/2} z_0^{d_n-1} z_n), \text{ and } e_0 = (1, 0, \dots, 0).$$

*(Beltrán-Shub) Given  $(f, \zeta) \in \mathcal{V}_d \setminus \Sigma'$ , there exists a path in the condition metric with endpoints  $(g, e_0)$  and  $(f, \zeta)$  with condition length*

$$L_K \leq 9nD^{3/2} + 2\sqrt{n} \ln \left( \frac{\mu_{\text{norm}}(f, \zeta)}{\sqrt{n}} \right).$$

Following such a path will give a complexity in the logarithm of the condition number instead of its square !



# Self-Convexity

Along a condition geodesic do we really avoid ill-conditionned problems ? We may reformulate this question in the following terms :

Given a geodesic  $F(t)$  in  $\mathcal{V}_d \setminus \Sigma'$ , is the maximum of  $\mu_{norm}(F_t)$  necessarily reached at the endpoints of the path ? That is:

Is  $\mu_{norm}(F_t)$  a **quasi-convex function** ?

A stronger property would be:

Is  $\mu_{norm}(F_t)$  a **convex function** ?

A much stronger property would be:

Is  $\mu_{norm}(F_t)$  a **log-convex function** ?

# Self-Convexity

Along a condition geodesic do we really avoid ill-conditioned problems ? We may reformulate this question in the following terms :

Given a geodesic  $F(t)$  in  $\mathcal{V}_d \setminus \Sigma'$ , is the maximum of  $\mu_{norm}(F_t)$  necessarily reached at the endpoints of the path ? That is:

Is  $\mu_{norm}(F_t)$  a **quasi-convex function** ?

A stronger property would be:

Is  $\mu_{norm}(F_t)$  a **convex function** ?

A much stronger property would be:

Is  $\mu_{norm}(F_t)$  a **log-convex function** ?

# Self-Convexity

Along a condition geodesic do we really avoid ill-conditioned problems ? We may reformulate this question in the following terms :

Given a geodesic  $F(t)$  in  $\mathcal{V}_d \setminus \Sigma'$ , is the maximum of  $\mu_{norm}(F_t)$  necessarily reached at the endpoints of the path ? That is:

Is  $\mu_{norm}(F_t)$  a **quasi-convex function** ?

A stronger property would be:

Is  $\mu_{norm}(F_t)$  a **convex function** ?

A much stronger property would be:

Is  $\mu_{norm}(F_t)$  a **log-convex function** ?

# Self-Convexity

Along a condition geodesic do we really avoid ill-conditioned problems ? We may reformulate this question in the following terms :

Given a geodesic  $F(t)$  in  $\mathcal{V}_d \setminus \Sigma'$ , is the maximum of  $\mu_{norm}(F_t)$  necessarily reached at the endpoints of the path ? That is:

Is  $\mu_{norm}(F_t)$  a **quasi-convex function** ?

A stronger property would be:

Is  $\mu_{norm}(F_t)$  a **convex function** ?

A much stronger property would be:

Is  $\mu_{norm}(F_t)$  a **log-convex function** ?

# Self-Convexity Bis

We call this last property **self-convexity of the condition number** that is convexity of the logarithm of the condition number in the condition Riemannian structure.

# The linear case

Self-convexity holds in the linear case, that is when all the degrees are equal to 1. Let us be more precise.

Let two integers  $1 \leq n \leq m$  be given and let us denote by  $\mathbb{GL}_{n,m}$  the space of matrices  $A \in \mathbb{K}^{n \times m}$  with maximal rank :  $\text{rank } A = n$ ,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ .

The singular values of such matrices are denoted in decreasing order:

$$\sigma_1(A) \geq \dots \geq \sigma_{n-1}(A) \geq \sigma_n(A) > 0.$$

# The linear case

Self-convexity holds in the linear case, that is when all the degrees are equal to 1. Let us be more precise.

Let two integers  $1 \leq n \leq m$  be given and let us denote by  $\mathbb{GL}_{n,m}$  the space of matrices  $A \in \mathbb{K}^{n \times m}$  with maximal rank :  $\text{rank } A = n$ ,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ .

The singular values of such matrices are denoted in decreasing order:

$$\sigma_1(A) \geq \dots \geq \sigma_{n-1}(A) \geq \sigma_n(A) > 0.$$

# The linear case

Self-convexity holds in the linear case, that is when all the degrees are equal to 1. Let us be more precise.

Let two integers  $1 \leq n \leq m$  be given and let us denote by  $\mathbb{GL}_{n,m}$  the space of matrices  $A \in \mathbb{K}^{n \times m}$  with maximal rank :  $\text{rank } A = n$ ,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ .

The singular values of such matrices are denoted in decreasing order:

$$\sigma_1(A) \geq \dots \geq \sigma_{n-1}(A) \geq \sigma_n(A) > 0.$$



# Self-convexity: the linear case bis

The smallest singular value  $\sigma_n(A)$  is a locally Lipschitz map in  $\mathrm{GL}_{n,m}$ .

It is smooth on the open subset

$$\mathrm{GL}_{n,m}^> = \{A \in \mathrm{GL}_{n,m} : \sigma_{n-1}(A) > \sigma_n(A)\}.$$

# Self-convexity: the linear case bis

The smallest singular value  $\sigma_n(A)$  is a locally Lipschitz map in  $\mathbb{GL}_{n,m}$ .

It is smooth on the open subset

$$\mathbb{GL}_{n,m}^> = \{A \in \mathbb{GL}_{n,m} : \sigma_{n-1}(A) > \sigma_n(A)\}.$$

# Self-convexity: the linear case

The length of the path  $X(t) \in \mathbb{GL}_{n,m}$  in condition metric is defined by the integral

$$L_{\kappa}(X) = \int_0^1 \left\| \frac{d}{dt} X(t) \right\|_F \sigma_n(X(t))^{-1} dt$$

where

$$\|M\|_F^2 = \sum |m_{ij}|^2$$

denotes the Frobenius norm of the matrix  $M$ .

# Self-convexity: the linear case quater

*(Beltrán-Dedieu-Malajovich-Shub)  $\sigma_n(X(t))^{-2}$  is logarithmically convex along the geodesics in  $\mathbb{GL}_{n,m}$  for the condition metric.*

Its proof is particularly long and difficult.

# Self-convexity: the linear case quater

*(Beltrán-Dedieu-Malajovich-Shub)  $\sigma_n(X(t))^{-2}$  is logarithmically convex along the geodesics in  $\mathbb{GL}_{n,m}$  for the condition metric.*

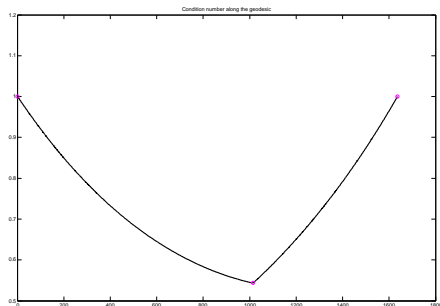
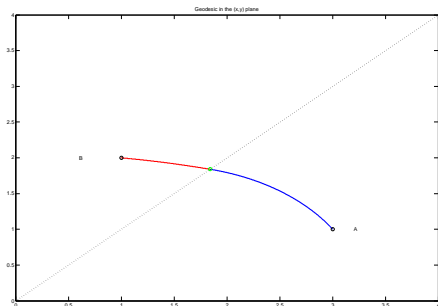
Its proof is particularly long and difficult.

## $2 \times 2$ diagonal case

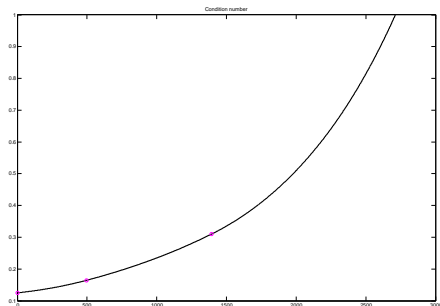
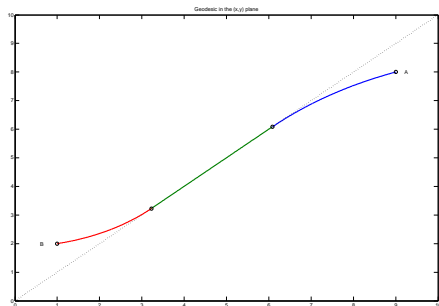
We consider, in the two following examples,  $2 \times 2$  real diagonal matrices with positive diagonal entries. The condition length is given by

$$L_{\kappa} = \int_a^b \frac{\sqrt{\dot{x}_1(t)^2 + \dot{x}_2(t)^2}}{\min(x_1(t), x_2(t))} dt.$$

# $2 \times 2$ diagonal case, bis



# $2 \times 2$ diagonal case, ter





# Self-convexity: extensions

Self-convexity holds in a slightly different context.

Let  $\mathcal{N}$  be a  $C^2$  submanifold without boundary in  $\mathbb{R}^j$ . Let us denote by

$$d(x, \mathcal{N}) = \min_{y \in \mathcal{N}} \|x - y\| \quad \text{and} \quad \alpha(x) = d(x, \mathcal{N})^{-2}.$$

Let  $\mathcal{U}$  be the largest open set in  $\mathbb{R}^j$  such that, for any  $x \in \mathcal{U}$  there is a unique closest point in  $\mathcal{N}$  to  $x$ . When  $\mathcal{U}$  is equipped with the new metric

$$\langle \cdot, \cdot \rangle_{\kappa, x} = \alpha(x) \langle \cdot, \cdot \rangle$$

we have

*(Beltrán-Dedieu-Malajovich-Shub)  $\alpha : \mathcal{U} \setminus \mathcal{N} \rightarrow \mathbb{R}$  is self-convex.*

# Self-convexity: extensions

Self-convexity holds in a slightly different context.

Let  $\mathcal{N}$  be a  $C^2$  submanifold without boundary in  $\mathbb{R}^j$ . Let us denote by

$$d(x, \mathcal{N}) = \min_{y \in \mathcal{N}} \|x - y\| \text{ and } \alpha(x) = d(x, \mathcal{N})^{-2}.$$

Let  $\mathcal{U}$  be the largest open set in  $\mathbb{R}^j$  such that, for any  $x \in \mathcal{U}$  there is a unique closest point in  $\mathcal{N}$  to  $x$ . When  $\mathcal{U}$  is equipped with the new metric

$$\langle \cdot, \cdot \rangle_{\kappa, x} = \alpha(x) \langle \cdot, \cdot \rangle$$

we have

*(Beltrán-Dedieu-Malajovich-Shub)  $\alpha : \mathcal{U} \setminus \mathcal{N} \rightarrow \mathbb{R}$  is self-convex.*

# Self-convexity: extensions

Self-convexity holds in a slightly different context.

Let  $\mathcal{N}$  be a  $C^2$  submanifold without boundary in  $\mathbb{R}^j$ . Let us denote by

$$d(x, \mathcal{N}) = \min_{y \in \mathcal{N}} \|x - y\| \text{ and } \alpha(x) = d(x, \mathcal{N})^{-2}.$$

Let  $\mathcal{U}$  be the largest open set in  $\mathbb{R}^j$  such that, for any  $x \in \mathcal{U}$  there is a unique closest point in  $\mathcal{N}$  to  $x$ . When  $\mathcal{U}$  is equipped with the new metric

$$\langle \cdot, \cdot \rangle_{\kappa, x} = \alpha(x) \langle \cdot, \cdot \rangle$$

we have

*(Beltrán-Dedieu-Malajovich-Shub)  $\alpha : \mathcal{U} \setminus \mathcal{N} \rightarrow \mathbb{R}$  is self-convex.*

# Some open problems

- 1 Does self-convexity hold in the case of polynomial systems ?
- 2 How to approximate minimizing condition geodesics in the solution variety ?
- 3 More generally, how to compute geodesics in a Lipschitz-Riemannian structure ?

# Some open problems

- 1 Does self-convexity hold in the case of polynomial systems ?
- 2 How to approximate minimizing condition geodesics in the solution variety ?
- 3 More generally, how to compute geodesics in a Lipschitz-Riemannian structure ?

# Some open problems

- 1 Does self-convexity hold in the case of polynomial systems ?
- 2 How to approximate minimizing condition geodesics in the solution variety ?
- 3 More generally, how to compute geodesics in a Lipschitz-Riemannian structure ?

# References

- ① M. SHUB, *Complexity of Bézout's Theorem VI: Geodesics in the Condition Metric*, FOCM 9 (2009) 171-178.
- ② C. BELTRÁN, AND M. SHUB, *Complexity of Bézout's Theorem VII: Distances Estimates in the Condition Metric*, FOCM 9 (2009) 179-195.
- ③ C. BELTRÁN, J.-P. DEDIEU, G. MALAJOVICH, AND M. SHUB, *Convexity properties of the condition number*. To appear in SIMAX.
- ④ P. BOITO, AND J.-P. DEDIEU, *The condition metric in the space of rectangular full rank matrices*. To appear in SIMAX.
- ⑤ D. ARMENTANO, *Complexity of path-following methods for the algebraic eigenvalue problem*. Preprint.