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An overview of triangularizability results on collections of compact operators

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Outline of the talk

In this talk, we consider collections of compact (resp. $\mathcal{C}_p$ class) operators on arbitrary Banach (resp. Hilbert) spaces. It is proved that a collection of triangularizable compact operators on a real Banach (resp. Hilbert) space is triangularizable if and only if the collection of induced operators on the Taylor (resp. natural) complexification of the real Banach (resp. Hilbert) space is triangularizable. In view of this result and some easy-to-check facts concerning the properties of induced operators acting on the Taylor (resp. natural) complexification of real Banach (Hilbert) spaces, we prove that every triangularizability result on certain collections of compact operators on a complex Banach (resp. Hilbert) space gives rise to its counterpart on a real Banach (resp. Hilbert) space. We use our main results to present new proofs as well as extensions of certain classical theorems (e.g., those due to Kolchin, McCoy, and others) on arbitrary Banach (resp. Hilbert) spaces. If time permits, I will show that the notion of simultaneous triangularization for collections of triangularizable compact operators on arbitrary Banach spaces remains intact under taking certain limit operations. I will then use this result to prove an interesting invariant subspace theorem.

0. Definitions and Notational Conventions

Unless otherwise stated,

- $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

- $\mathcal{X}$: A separable real or complex Banach space.

• By a subspace of $\mathcal{X}$ we always mean a closed subspace of it. The subspaces $\{0\}$ and $\mathcal{X}$ are called the trivial subspaces of $\mathcal{X}$.

• We use $\mathcal{B}(\mathcal{X})$ (resp. $\mathcal{B}_0(\mathcal{X})$, $\mathcal{B}_{00}(\mathcal{X})$) to denote the set (in fact the ideal) of bounded operators (resp. compact operators, finite-rank operators) on $\mathcal{X}$.

• $\mathcal{V}$: A finite-dimensional vector space over a field F.

• We use $\mathcal{L}(\mathcal{V})$ to denote the set (in fact the algebra) of linear transformations on $\mathcal{V}$. Plainly, $\mathcal{L}(\mathcal{X}) = \mathcal{B}(\mathcal{X}) = \mathcal{B}_0(\mathcal{X})$ provided $\mathcal{X}$ is finite-dimensional.

• For a collection $\mathcal{F}$ of bounded operators (resp. linear transformations), we use $\mathcal{F}'$ to denote the commutant of $\mathcal{F}$; more precisely, $\mathcal{F}' := \{T \in \mathcal{B}(\mathcal{X}) : ST = TS \text{ for all } S \in \mathcal{F}\}$.

• A subspace $\mathcal{M}$ is *invariant* for a collection $\mathcal{F}$ of bounded operators (resp. linear transformations) if $T\mathcal{M} \subseteq \mathcal{M}$ for all $T \in \mathcal{F}$; $\mathcal{M}$ is *hyperinvariant* for a collection $\mathcal{F}$ of bounded operators if $T\mathcal{M} \subseteq \mathcal{M}$ for all $T \in \mathcal{F} \cup \mathcal{F}'$.

• A collection $\mathcal{F}$ of bounded operators (resp. linear transformations) is called *reducible* if $\mathcal{F} = \{0\}$ or it has a nontrivial invariant subspace. This definition is slightly unconventional, but it simplifies some of the statements in what follows.

• Irreducible $\equiv$ NOT reducible.

• A collection $\mathcal{F}$ of operators (resp. linear transformations) is called *simultaneously triangularizable* or simply *triangularizable* if there exists a maximal chain of subspaces of $\mathcal{X}$ each of which is invariant for $\mathcal{F}$. In case the underlying space is finite-dimensional, it is easily seen that triangularizability of a family of linear transformations is equivalent to the existence of a basis for the vector space such that all transformations in the family have upper triangular matrix representation with respect to that basis.

• Let R be a subring of a field F. By an *R-algebra* $\mathcal{A}$ in $\mathcal{L}(\mathcal{V})$ (resp. $M_n(F)$), we mean a subring of $\mathcal{L}(\mathcal{V})$ (resp. $M_n(F)$) that is closed under scalar multiplication by the elements of the subring R. For a collection $\mathcal{C}$ in $\mathcal{L}(\mathcal{V})$ (resp. $M_n(F)$), we use $\mathrm{Alg}_R(\mathcal{C})$ to denote the R-algebra generated by $\mathcal{C}$. By $\mathrm{Alg}(\mathcal{C})$ we simply mean $\mathrm{Alg}_F(\mathcal{C})$.

• Plainly, a family $\mathcal{F}$ of linear operators (resp. linear transformations) is triangularizable iff $\mathrm{Sem}(\mathcal{F})$, *the semigroup generated by* $\mathcal{F}$, is triangularizable; or iff $\mathrm{Alg}(\mathcal{F})$, *the algebra generated by* $\mathcal{F}$, is triangularizable. Also, for every family $\mathcal{F}$ of bounded operators (resp. linear transformations)

$$\mathcal{F}' = (\mathrm{Alg}(\mathcal{F}))' = (\mathrm{Sem}(\mathcal{F}))'.$$

Thus $\mathcal{F}$ has a non-trivial hyperinvariant subspace iff $\mathrm{Sem}(\mathcal{F})$ does, or iff $\mathrm{Alg}(\mathcal{F})$ does.

• If $\mathcal{S}$ is a multiplicative semigroup, a subset $\mathcal{J}$ of $\mathcal{S}$ is called a *semigroup ideal* of $\mathcal{S}$ if $JS, SJ \in \mathcal{J}$ whenever $J \in \mathcal{J}$ and $S \in \mathcal{S}$.

• Let $\mathcal{X}$ be a real vector space. We use the symbol $\widetilde{\mathcal{X}}$ to denote $\mathcal{X} \times \mathcal{X}$, *the algebraic complexification of* $\mathcal{X}$, whose construction resembles that of $\mathbb{C}$ from $\mathbb{R}$, as follows

$$(x, y) + (u, v) := (x + u, y + v), (a + ib)(x, y) := (ax - by, bx + ay),$$

where $x, y, u, v, \in \mathcal{X}$ and $a, b \in \mathbb{R}$. It is easily verified that $\widetilde{\mathcal{X}}$ is a complex vector space into which $\mathcal{X}$ embeds via the mapping $x \to (x, 0)$. With that in mind, we can use the

familiar notation $z = x + iy$ to denote the vector $z = (x, y)$ in $\widetilde{\mathcal{X}}$. Thus, if $z = x + iy$, it is natural to define $\mathrm{Re}(z) := x$ and $\mathrm{Im}(z) := y$. Also, in a natural way, by *the conjugate* of an element $z = x + iy$ of $\widetilde{\mathcal{X}}$, we mean the element $\bar{z}$ defined by $\bar{z} := x - iy$.

• A norm $||.||_{\widetilde{\mathcal{X}}}$ on $\widetilde{\mathcal{X}}$ is called a *reasonable complexification norm* provided that

$$||\mathrm{Re}(z)||_{\widetilde{\mathcal{X}}} = ||\mathrm{Re}(z)||, \;\; ||\bar{z}||_{\widetilde{\mathcal{X}}} = ||z||_{\widetilde{\mathcal{X}}},$$

for all $z \in \widetilde{\mathcal{X}}$, where $||.||$ denotes the norm of $\mathcal{X}$. It is not difficult to check that the norm $||.||_T$, called *the Taylor [complexification] norm* of $\widetilde{\mathcal{X}}$, defined by

$$||x + iy||_T := \sup_{a^2+b^2=1} ||ax + by||,$$

where $x, y \in \mathcal{X}$, is in fact the smallest reasonable complexification norm on $\widetilde{\mathcal{X}}$. Let $\mathcal{H}$ be a real Hilbert space. It is easily seen that the norm defined by

$$||x + iy|| := (||x||^2 + ||y||^2)^{1/2},$$

is a reasonable complexification norm on $\widetilde{\mathcal{H}}$ which comes from the following inner product on $\widetilde{\mathcal{H}}$

$$\langle x + iy, u + iv\rangle := \langle x, u\rangle + i\langle y, u\rangle - i\langle x, v\rangle + \langle y, v\rangle.$$

We call $\widetilde{\mathcal{H}}$ *the natural complexification* of $\mathcal{H}$.

1. Some elementary and preliminary lemmas

• Lemma 1.1.

(i) *Let $\mathcal{V}$ be a finite-dimensional vector space over a field F, and $\mathcal{S}$ a semigroup in $\mathcal{L}(\mathcal{V})$. If $\mathcal{S}$ is irreducible, then so is every nonzero semigroup ideal of $\mathcal{S}$.*

(ii) *Let $\mathcal{X}$ be a real or complex Banach space, and $\mathcal{S}$ a semigroup in $\mathcal{B}(\mathcal{X})$. If $\mathcal{S}$ is irreducible, then so is every nonzero semigroup ideal of $\mathcal{S}$.*

If $\mathcal{V}$ is a vector space and $\mathcal{N}$ is a subspace of $\mathcal{V}$, then the *quotient space* **$\mathcal{V}/\mathcal{N}$ is the collection of cosets $[x] = x + \mathcal{N} = \{x + z : z \in \mathcal{N}\}$ for $x \in \mathcal{V}$, with $[x] + [y]$ defined as $[x + y]$ and $\lambda[x]$ defined as $[\lambda x]$ for scalars λ. If A is a linear transformation on $\mathcal{V}$ and $\mathcal{N}$ is invariant under A, then the quotient transformation $\hat{A}$ on $\mathcal{V}/\mathcal{N}$ is defined by $\hat{A}[x] = [Ax]$ for each $x \in \mathcal{V}$; (the invariance of $\mathcal{N}$ under A ensures that $\hat{A}$ is well-defined on the cosets). If $\mathcal{C}$ is a collection of linear transformations on $\mathcal{V}$, and if $\mathcal{M}$ and $\mathcal{N}$ are invariant subspaces for $\mathcal{C}$ with $\mathcal{N}$ properly contained in $\mathcal{M}$, then the** *collection of quotients of $\mathcal{C}$ with respect to $\{\mathcal{M}, \mathcal{N}\}$* **is the set of all quotient transformations $\hat{A}$ on $\mathcal{M}/\mathcal{N}$. We say that a property $\mathcal{P}$ is** *inherited by quotients* **if every collection of quotients of a collection satisfying $\mathcal{P}$ also satisfies $\mathcal{P}$.**

• **Lemma 1.2. (The Triangularization Lemma)**
Let $\mathcal{P}$ be a set of properties of families of linear operators (resp. linear transformations) each of which is inherited by quotients. If every family of operators (resp. transformations) on a space of dimension greater than one that satisfies $\mathcal{P}$ is reducible, then every family satisfying $\mathcal{P}$ is triangularizable.

In what follows, we shall make frequent use of the following useful lemma.

• **Lemma 1.3.**

(i) *Let $\mathcal{V}$ be a finite-dimensional vector space over a field F, $\mathcal{S}$ a semigroup in $\mathcal{L}(\mathcal{V})$, and T a nonzero linear transformation in $\mathcal{L}(\mathcal{V})$. If $\mathcal{S}$ is irreducible, then so is $T\mathcal{S}|_{\mathcal{R}}$ where $\mathcal{R} = T\mathcal{V}$ is the range of T.*

(ii) *Let $\mathcal{X}$ be a real or complex Banach space, $\mathcal{S}$ a semigroup in $\mathcal{B}(\mathcal{X})$, and T a nonzero linear operator in $\mathcal{B}(\mathcal{X})$. If $\mathcal{S}$ is irreducible, then so is $T\mathcal{S}|_{\mathcal{R}}$ where $\mathcal{R} = \overline{T\mathcal{X}}$ is the closure of the range of T.*

From this point on, unless otherwise stated, the symbol $\widetilde{\mathcal{X}}$ stands for the Taylor complexification of the real Banach space $\mathcal{X}$, and $\widetilde{\mathcal{H}}$ for the natural complexification of the real Hilbert space $\mathcal{H}$. It is easily checked that $\lim_n(x_n + iy_n) = x + iy$ in $\widetilde{\mathcal{X}}$ (resp. $\widetilde{\mathcal{H}}$) iff $\lim_n x_n = x$ and $\lim_n y_n = y$ in $\mathcal{X}$ (resp. in $\mathcal{H}$). Therefore, $\widetilde{\mathcal{X}}$ (resp. $\widetilde{\mathcal{H}}$) is a complex Banach (resp. Hilbert) space. If $T \in \mathcal{B}(\mathcal{X})$ (resp. $T \in \mathcal{B}(\mathcal{H})$), then the operator $\widetilde{T} \in \mathcal{B}(\widetilde{\mathcal{X}})$ (resp. $\widetilde{T} \in \mathcal{B}(\widetilde{\mathcal{H}})$) defined by $\widetilde{T}(x + iy) := Tx + iTy$ is a bounded operator and furthermore $||\widetilde{T}|| = ||T||$. As a matter of fact, the tilde is a covariant functor from the category of real Banach (resp. Hilbert) spaces into the category of complex Banach (resp. Hilbert) spaces.

The following proposition gathers some straightforward facts concerning some properties of the Taylor (resp. natural) complexification of real Banach (resp. Hilbert) spaces and operators acting on them which will be needed in what follows.

• Proposition 1.4.

Let $\mathcal{X}$ (resp. $\mathcal{H}$) be a real Banach (resp. Hilbert) space, $T \in \mathcal{B}(\mathcal{X})$ (resp. $T \in \mathcal{B}(\mathcal{H})$), and $\widetilde{T}$ its extension to $\widetilde{\mathcal{X}}$ (resp. $\widetilde{\mathcal{H}}$). Then the following hold.

(i) $\operatorname{rank}(T) = \operatorname{rank}(\widetilde{T})$ *and* $\dim\ker(T) = \dim\ker\widetilde{T}$.

(ii) The operator T is compact iff $\widetilde{T}$ is compact.

(iii) If $\mathcal{M}$ is a subspace of $\mathcal{X}$ (resp. $\mathcal{H}$), then $\mathcal{M}+i\mathcal{M}$ is a subspace of $\widetilde{\mathcal{X}}$ (resp. $\widetilde{\mathcal{H}}$). Conversely, if $\mathcal{M}$ is a subset of $\mathcal{X}$ (resp. $\mathcal{H}$) such that $\mathcal{M}+i\mathcal{M}$ is a subspace of $\widetilde{\mathcal{X}}$ (resp. $\widetilde{\mathcal{H}}$), then $\mathcal{M}$ is a subspace of $\mathcal{X}$ (resp. $\mathcal{H}$).

(iv) A chain $\mathcal{C} = \{\mathcal{M}\}_{\mathcal{M}\in\mathcal{C}}$ is a maximal chain of subspaces of $\mathcal{X}$ iff $\widetilde{\mathcal{C}} := \{\mathcal{M}+i\mathcal{M}\}_{\mathcal{M}\in\mathcal{C}}$ is a maximal chain of subspaces for $\widetilde{\mathcal{X}}$.

(v) The bounded operator T is a $\mathcal{C}_p$ class operator on $\mathcal{H}$ iff $\widetilde{T}$ is a $\mathcal{C}_p$ class operator on $\widetilde{\mathcal{H}}$. Moreover, $\operatorname{tr}(T) = \operatorname{tr}(\widetilde{T})$ *provided that T is a trace class operator on $\mathcal{H}$.*

(vi) A subspace $\mathcal{M}$ is invariant under T iff the subspace $\mathcal{M}+i\mathcal{M}$ is invariant under $\widetilde{T}$.

(vii) If T is compact and triangularizable, then $\sigma(T) = \sigma(\widetilde{T})$. Also, if T is compact and $\sigma(\widetilde{T}) \subset \mathbb{R}$, then T is triangularizable and $\sigma(T) = \sigma(\widetilde{T})$. Moreover, the compact triangularizable operators T and $\widetilde{T}$ share the same set of eigenvalues counting multiplicity.

2. Main results

• **Lemma 2.1**

Let $\mathcal{X}$ (resp. $\mathcal{H}$) be a real Banach (resp. Hilbert) space, $\mathcal{F}$ a family of triangularizable compact operators on $\mathcal{X}$ (resp. $\mathcal{H}$). Then $\mathcal{F}$ is triangularizable over $\mathcal{X}$ (resp. $\mathcal{H}$) iff the family $\widetilde{\mathcal{F}}$, the family *consisting of the extensions of the members of* $\mathcal{F}$ *to* $\widetilde{\mathcal{X}}$ *(resp.* $\widetilde{\mathcal{H}}$*), is triangularizable over* $\widetilde{\mathcal{X}}$ *(resp.* $\widetilde{\mathcal{H}}$*).*

With the preceding lemma at our disposal, we are now ready to prove that every triangularizability result on certain collections of compact operators acting on a complex Banach space gives rise to its counterpart on the same certain collections of triangularizable compact operators acting on a real Banach space. To state the result, we need some definitions. Let $\mathcal{P}$ be a set of properties of operators, e.g., properties of the rank, the nullity, the trace, the spectrum , and/or the spectral radius of operators. The property $\mathcal{P}$ is said to be *admissible* if the operator $\widetilde{T}$ satisfies the property $\mathcal{P}$ whenever the operator T does. For instance, in view of Proposition 1.4, properties of the rank, the nullity, the trace, and the spectrum of operators are admissible properties of compact (trace class) triangularizable operators. An operator satisfying a set of properties, say $\mathcal{P}$, of operators is called a $\mathcal{P}$*-operator.* Let $\mathcal{Q}$ be a set of properties of collections of operators, e.g., commutativity, consisting of (quasi)nilpotent operators, being closed under multiplication (this would give rise to the notion of semigroups of operators), etc. The property $\mathcal{Q}$ is called *admissible* if for each family $\mathcal{F}$ of operators in $\mathcal{B}(\mathcal{X})$, the family $\widetilde{\mathcal{F}} = \{\widetilde{T} : T \in \mathcal{F}\}$ satisfies $\mathcal{Q}$ whenever the family $\mathcal{F}$ does. Roughly speaking, a set of properties of operators (resp. collections of operators) is admissible if it is preserved under taking the tilde operation. A family that satisfies a property $\mathcal{Q}$ of collections of operators is called a $\mathcal{Q}$*-family of operators.* We note that every operator satisfies the property $\emptyset$, i.e., the empty property, and that every family of operators is an $\emptyset$-family of operators! Moreover, the empty property is an admissible set of properties of operators (resp. collections of operators)!

• Proposition 2.2.
Let $\mathcal{P}$ and $\mathcal{Q}$ be sets of admissible properties of operators and collections of operators, respectively. If every $\mathcal{Q}$-family of compact $\mathcal{P}$-operators on a complex Banach (resp. Hilbert) space is triangularizable, then so is every $\mathcal{Q}$-family consisting of triangularizable compact $\mathcal{P}$-operators acting on a real Banach (resp. Hilbert) space.

Let $\mathcal{X}$ be a complex (resp. real) Banach space, and S a subset of $\mathbb{C}$ (resp. $\mathbb{R}$). By an *S-semigroup* $\mathcal{S}$ of $\mathcal{B}(\mathcal{X})$, we mean a multiplicative semigroup of $\mathcal{S}$ bounded operators that is closed under scalar multiplication by the elements of S.

• Lemma 2.1 (Radjavi).
Let $\mathcal{X}$ be a complex (resp. real) Banach space and let $\mathcal{S}$ be a uniformly closed $\mathbb{R}^+$-semigroup of compact triangularizable operators on $\mathcal{X}$ where $\mathbb{R}^+$ denotes the set of positive real numbers. If $\mathcal{S}$ contains an operator that is not quasinilpotent, then $\mathcal{S}$ contains a nonzero finite-rank operator that is either idempotent or nilpotent.

• Theorem 2.2.
Let $\mathcal{X}$ be a complex (resp. real) Banach space of dimension greater than 1, R a subring of $\mathbb{R}$, and $\mathcal{A}$ an R-algebra of triangularizable compact operators on $\mathcal{X}$ with spectra in R. Then $\mathcal{A}$ is reducible.

• Corollary 2.3.
Let $\mathcal{X}$ be a complex (resp. real) Banach space, R a subring of $\mathbb{R}$, and $\mathcal{A}$ an R-algebra of compact operators on $\mathcal{X}$ with spectra in R. Then $\mathcal{A}$ is triangularizable iff every element of $\mathcal{A}$ is triangularizable.

Part (i) of the following lemma is from Radjavi. Lemmas 2.4(ii), 2.5(ii), and 2.5(iv) below are slight generalizations of results due to Radjavi.

• Lemma 2.4.

(i) *Let* $\sum_{i=1}^{\infty} a_i$ *be an absolutely convergent series in* $\mathbb{C}$ *with* $|a_i| < 1$ *for all* $i \in \mathbb{N}$. *Then*

$$\lim_n \sum_{i=1}^{\infty} a_i^n = 0.$$

(ii) *Let* $a_i \in \mathbb{C}$ *with* $|a_i| = 1$ $(1 \le i \le m)$ *be such that* $\lim_n \sum_{i=1}^{m} a_i^n = c$ *where* $c \in \mathbb{C}$. *Then* $c = m$ *and* $a_i = 1$ *for all* $1 \le i \le m$.

(iii) *Let* $a_i, b_j \in \mathbb{C}$ *with* $|a_i| = |b_j| = 1$ $(1 \le i \le m, 1 \le j \le n)$ *be such that* $\lim_k (\sum_{i=1}^{m} a_i^k - \sum_{j=1}^{n} b_j^k) = 0$. *Then* $m = n$ *and there is a permutation* σ *on* m *letters such that* $b_i = a_{\sigma(i)}$ *for all* $1 \le i \le m$.

• Lemma 2.5.

Let $\sum_{j=1}^{\infty} \lambda_j$ *and* $\sum_{j=1}^{\infty} \mu_j$ *be two absolutely convergent series in* $\mathbb{C}$, *and let* $m \in \mathbb{N}$ *be given.*

(i) *If* $\lambda_j, \mu_j \in \mathbb{C} \setminus \{0\}$ *for all* $j \in \mathbb{N}$, *and*

$$\sum_{j=1}^{\infty} \lambda_j^k = \sum_{j=1}^{\infty} \mu_j^k,$$

for all $k \in \mathbb{N}$ *with* $k \ge m$, *then there is a permutation* σ *on* $\mathbb{N}$ *such that* $\mu_j = \lambda_{\sigma(j)}$.

(ii) *If for some* $C \in \mathbb{C}$

$$\sum_{j=1}^{\infty} \lambda_j^k = C,$$

for all $k \in \mathbb{N}$ *with* $k \ge m$, *then* C *is a nonnegative integer and* $\lambda_j = 0$ *or* 1 *for all* $j \in \mathbb{N}$.

(iii) *If for some $c \in \mathbb{C}$*

$$\sum_{j=1}^{\infty} \lambda_j^k = c^{k-m} \sum_{j=1}^{\infty} \lambda_j^m,$$

for all $k \in \mathbb{N}$ with $k \geq m$, then $\lambda_j = 0$ or c for all $j \in \mathbb{N}$.

(iv) *Let $\mathcal{H}$ be a real or complex Hilbert space, and $A \in \mathcal{C}_p(\mathcal{H})$. Then A is quasinilpotent iff*

$$\operatorname{tr}(A^k) = 0,$$

for each $k \in \mathbb{N}$ with $k \geq m$ where $m \in \mathbb{N}$ with $m > p$.

Recall that a semigroup (resp. algebra) of compact quasinilpotent operators on a Banach space is called a *Volterra semigroup* **(resp.** *Volterra algebra***). Here we give a new proof of the following well-known theorem which is due to Radjavi extending Kaplansky's Theorem to trace class operators. It is worth mentioning that the original theorem was established for complex Hilbert spaces only.**

• **Theorem 2.6.**

Let $\mathcal{H}$ be a real or complex Hilbert space, $\mathcal{S}$ a semigroup in $\mathcal{C}_1$ on which trace is constant. Then the semigroup $\mathcal{S}$ is triangularizable. In particular, if trace is zero on a semigroup $\mathcal{S}$ in $\mathcal{C}_1$, then the algebra generated by $\mathcal{S}$ is a Volterra algebra of $\mathcal{C}_1$ operators.

Proof. In view of Corollary 2.3, it suffices to show that the $\mathbb{R}$-algebra $\mathcal{A}$ generated by $\mathcal{S}$ consists of triangularizable operators with spectra in $\mathbb{R}$. To this end, suppose that $\operatorname{tr}(\mathcal{S}) = \{C\}$ for some $C \in \mathbb{F}$. Suppose that $A = c_1S_1 + ... + c_kS_k \in \mathcal{A}$ where $k \in \mathbb{N}$, $c_j \in \mathbb{R}$, $S_j \in S$ for each $j = 1, ..., k$ is given. Since $\operatorname{tr}(\mathcal{S}) = \{C\}$, it is easily seen that $\operatorname{tr}(A^j) = C(c_1 + ... + c_k)^j$ for all $j \in \mathbb{N}$. If $c_1 + ... + c_k = 0$ it follows from Lemma 2.5(iv) that A is quasinilpotent and hence it is triangularizable. If $c := c_1 + ... + c_k \neq 0$, then $\operatorname{tr}(\frac{A}{c})^j = C$ for all $j \in \mathbb{N}$. So it follows from Lemma 2.5(ii), C is an integer, A/c is triangularizable, and that $\sigma(\frac{A}{c}) \subset \{0, 1\}$. Hence A is triangularizable and $\sigma(A) \subset \{0, c\}$ where $A = c_1S_1 + ... + c_kS_k \in \mathcal{A}$ and $c := c_1 + ... + c_k \in \mathbb{R}$ (In particular, $\sigma(S) \subset \{0, 1\}$ for all $S \in \mathcal{S}$). Thus $\mathcal{A}$, and hence $\mathcal{S}$, is triangularizable. For the rest, in view of the preceding lemma, it is easily seen that the algebra $\mathcal{A}$ generated by $\mathcal{S}$ is indeed a Volterra algebra of $\mathcal{C}_1$ operators. □

The following is a quick consequence of the preceding lemma.

• **Corollary 2.7.**
Let $\mathcal{H}$ be an arbitrary Hilbert space. If an algebra $\mathcal{A}$ in $\mathcal{C}_1$ is spanned by its quasinilpotent members as a vector subspace of $\mathcal{C}_1$, then the algebra $\mathcal{A}$ is a Volterra algebra of $\mathcal{C}_1$ operators, and therefore it is triangularizable.

Proof. Just note that the trace is zero on the algebra $\mathcal{A}$, hence the preceding lemma applies. $\square$

Remarks.

1. By results of Fong and Sourour every compact (resp. Hilbert-Schmidt, i.e., $\mathcal{C}_2$) operator on an infinite dimensional Hilbert space is a sum of two compact (resp. Hilbert-Schmidt) quasinilpotent operators. This would imply that the ideal of compact (resp. Hilbert-Schmidt) operators on an infinite dimensional Hilbert space, which is obviously irreducible, as a vector space, is spanned by its quasinilpotent members. Therefore, the preceding corollary cannot be generalized to algebras of compact (resp. $\mathcal{C}_p$, $p > 1$) operators on infinite dimensional Banach (resp. Hilbert) spaces.
2. A proof almost identical to that of the corollary (resp. the preceding lemma) shows that the counterpart of the corollary (resp. the preceding theorem) holds for algebras (resp. semigroups) of finite rank operators on an arbitrary Banach space.

A consequence of the preceding corollary is the following which can be thought of as a generalization of Kolchin's Theorem to $\mathcal{C}_1$ class operators on a real or complex Hilbert space.

- **Corollary 2.8.**

(i) Let $\mathcal{H}$ be a real or complex Hilbert space, $\mathcal{F}$ a family of $\mathcal{C}_1$ operators on $\mathcal{H}$ with the following properties: (a) every $A \in \mathcal{F}$ has trace zero (resp. can be written as a linear combination of quasinilpotent elements from the algebra generated by $\mathcal{F}$); (b) if A and B are in $\mathcal{F}$, then $AB + A + B$ is in $\mathcal{F}$. Then $\mathcal{F}$ is triangularizable.

(ii) Let $\mathcal{H}$ be a real or complex Hilbert space and $\mathcal{F}$ be a family of $\mathcal{C}_1$ operators on $\mathcal{H}$ such that every A in $\mathcal{F}$ has trace zero (resp. can be written as a linear combination of quasinilpotent elements from the algebra generated by $\mathcal{F}$). Then, every semigroup of operators of the form $I + Q$ with $Q \in \mathcal{F}$ is triangularizable.

Remarks.

1. A proof identical to that of the corollary shows that the counterpart of the corollary holds for collections of finite rank operators on an arbitrary Banach space and for collections of matrices in $M_n(F)$ where F is a field whose characteristic is zero or greater than n.

2. The proof of the corollary together with Radjavi's Trace Theorem implies the following generalization of Kolchin' Theorem in finite dimensions. *Let $n \in \mathbb{N}$, F a field with* $\mathrm{ch}(F) > n/2$ *or* $= 0$, *and $\mathcal{F}$ a family of triangularizable matrices in $M_n(F)$ with trace zero. Then, every semigroup of matrices of the form $I + A$ with $A \in \mathcal{F}$ is triangularizable.*

- **Theorem 2.9.**

Let $\mathcal{H}$ be a real or complex Hilbert space, $\mathcal{S}$ an irreducible semigroup of $\mathcal{C}_1$ operators, and $\mathcal{J}$ a nonzero semigroup ideal of $\mathcal{S}$. Then

(i)
$$\{A \in \mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\}) : \mathrm{tr}(A\mathcal{J}) = \{0\}\} = \{0\}.$$

(ii)
$$\{A \in \mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\}) : \rho(A\mathcal{J}) = \{0\}\} = \{0\}.$$

• Theorem 2.10.

Let $\mathcal{H}$ be a real or complex Hilbert space, $\mathcal{S}$ an irreducible semigroup of compact operators with $p \geq 1$, and $\mathcal{J}$ a nonzero semigroup ideal of $\mathcal{S}$. Then

(i)
$$\{A \in \mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\}) : \rho(\mathcal{J}A\mathcal{J}) = 0\} = \{0\}.$$

(ii)
$$\{A \in \mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\}) : \rho(A\mathcal{J}) = 0\} = \{0\}.$$

The following extends Guralnick's Theorem, which is, itself, an extension of a well-known theorem of McCoy, to compact operators (resp. $\mathcal{C}_p$ operators ($p \geq 1$)) on a real or complex Banach (resp. Hilbert) space.

• Theorem 2.11.

(i) Let $\mathcal{X}$ be a real or complex Banach space, $\mathcal{C}$ a collection of compact triangularizable operators, and $m \in \mathbb{N}$. Then $\mathcal{C}$ is triangularizable iff $(AB - BA)C$ is quasinilpotent for all $A, B \in \mathcal{C}$ and $C \in (\mathrm{Sem}(\mathcal{C}))^m$.

(ii) Let $\mathcal{H}$ be a real or complex Hilbert space, $\mathcal{C}$ a collection of triangularizable $\mathcal{C}_p$ operators with $p \geq 1$, and $m \in \mathbb{N}$ with $m > p$. Then $\mathcal{C}$ is triangularizable iff $\mathrm{tr}((AB - BA)C) = 0$ for all $A, B \in \mathcal{C}$ and $C \in (\mathrm{Sem}(\mathcal{C}))^m$.

We now use the Theorem 2.9 to prove the following result which is a slight generalization of Radjavi's Trace Theorem. Although, in light of Theorem 2.6, the proof presented here is standard but it applies to both real and complex Hilbert spaces and it is different from the original proof given by Radjavi.

• Theorem 2.12 (Radjavi's Trace Theorem).
Let $\mathcal{H}$ be a real or complex Hilbert space, and $\mathcal{F}$ a family of triangularizable $\mathcal{C}_p$ operators with $p \geq 1$. Then $\mathcal{F}$ is triangularizable if and only if trace is permutable on $\mathcal{F}^m$ for some integer $m \geq p$.

In finite dimensions, over general fields, Kaplansky showed that a semigroup of the form scalar plus nilpotent is triangularizable. In infinite dimensions, over complex Hilbert spaces, Nordgren-Radjavi-Rosenthal showed that a stronger result holds as follows. Below we give a new proof of the stronger result which works on both real and complex Banach spaces. It is worth mentioning that the result below does not hold in finite dimensions (e.g., if $n > 1$, and F is a field such that $\mathrm{ch}(F) = 0$ or $\mathrm{ch}(F)$ is not a divisor of n, then every matrix in $M_n(F)$ can be written as $\alpha I + N$ where N is a matrix with $\mathrm{tr}(N) = 0$).

• Theorem 2.13 (Nordgren-Radjavi-Rosenthal).
Let $\mathcal{H}$ be an infinite-dimensional real or complex Hilbert space. Then every semigroup $\mathcal{S}$ of operators of the form $\alpha I + N$ where N is a trace class operator with $\mathrm{tr}(N) = 0$ and with $\alpha \in \mathbb{F}$ is triangularizable.

• Theorem 2.14
Let $\mathcal{X}$ be an arbitrary Banach space, $\mathcal{F}_i, \mathcal{F}$ $(i \in \mathbb{N})$ nonempty families of compact operators on $\mathcal{X}$ such that each family $\mathcal{F}_n$ $(n \in \mathbb{N})$ is triangularizable and that $\lim_n \mathrm{dist}(\mathcal{F}_n, f) = 0$ for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is triangularizable.

• Theorem 2.15.
Let $\mathcal{X}$ be a real or complex Banach space, $A_n, A \in \mathcal{B}(\mathcal{X})$, and $K_n, K \in \overline{\mathcal{B}_{00}(\mathcal{X})}$ $(n \in \mathbb{N})$ with $\mathrm{rank}(K) \geq 2$. If $\text{s-}\lim_n A_n = A$, $\lim_n K_n = K$, and $\{A_n, K_n\}$ is triangularizable for each $n \in \mathbb{N}$, then A has a nontrivial invariant subspace.

• Corollary 2.16.

(i) Let $\mathcal{X}$ *be a real or complex Banach space,* $A_n, A \in \mathcal{B}(\mathcal{X})$, *and* $K \in \overline{\mathcal{B}_{00}(\mathcal{X})}$ $(n \in \mathbb{N})$ *with* $\operatorname{rank}(K) \geq 2$. *If* $\text{s-lim}_n A_n = A$, *and* $\{A_n, K\}$ *is triangularizable for each* $n \in \mathbb{N}$, *then* A *has a nontrivial invariant subspace.*

(ii) Let $\mathcal{H}$ *be a real or complex Hilbert space,* $(\alpha_i)_{i \in \mathbb{N}}$ *be an orthonormal basis for* $\mathcal{H}$, *and* $A_n, A \in \mathcal{B}(\mathcal{H})$. *If* $\text{s-lim}_n A_n = A$, *and for each* $n \in \mathbb{N}$ *there exists a permutation* π_n *on* $\mathbb{N}$ *such that* $(\alpha_{\pi_n(i)})_{i \in \mathbb{N}}$ *is a triangularizing chain for* A_n, *then* A *has a nontrivial invariant subspace.*