

# On the Taylor Spectrum of n-tuples of Toeplitz Operators on the Polydisk

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Multivariable Operator Theory Workshop  
Fields Institute  
August 2009

- brief introduction to  $\sigma_T$
- Thm  $\sigma_T((T_{f_1}, \dots, T_{f_m}), \mathcal{H}) = \overline{\text{ran } F(D^n)}^{\mathbb{C}^m}$
- algebraic ideas
- applications of these algebraic ideas
- open problems

$\mathcal{H}$  Hilbert space

(ex:  $\mathcal{H} = H^2(T^n)$ )

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not invertible in } B(\mathcal{H})\}$$

$$[0 \rightarrow \mathcal{H} \xrightarrow{T-\lambda I} \mathcal{H} \rightarrow 0 \text{ not exact}]$$

- (a)  $\sigma(T)$  compact,  $\neq \emptyset$  in  $\mathbb{C}$
- (b)  $p$  polynomial,  $\sigma(p(T)) = p(\sigma(T))$

$$T = (T_1, \dots, T_m) \quad T_i T_j = T_j T_i \text{ in } B(\mathcal{H})$$

" $\sigma(T_1, \dots, T_m)$ " such that

- (a)  $\sigma(T_1, \dots, T_m)$  compact,  $\neq \emptyset$  in  $\mathbb{C}^m$
- (b)  $p = (p_1, \dots, p_k) : \mathbb{C}^n \rightarrow \mathbb{C}^k$ ,  $p_i$  polynomials  
 $\sigma(p_1(T), \dots, p_k(T)) = p(\sigma(T_1, \dots, T_m))$
- (c)  $\sigma(T_1, \dots, T_s) = \text{Proj}_{\mathbb{C}^s}(\sigma(T_1, \dots, T_m))$   
all permutations

(Taylor, Vasilescu, Curto, ...)

(ref: [Curto](#))

$\Lambda_n[e]$  exterior algebra on  $n$  generators

$$e_1, \dots, e_n \quad (e_0 = 1) \quad e_i \wedge e_j + e_j \wedge e_i = 0$$

$$E_i : \Lambda_n[e] \rightarrow \Lambda_n[e] \quad E_i x = e_i \wedge x, \quad \text{so}$$

$$E_i E_j + E_j E_i = 0$$

$$T = (T_1, \dots, T_m) \quad T_i T_j = T_j T_i \quad \text{in } B(\mathcal{H})$$

$$D_T \in B(\mathcal{H} \otimes \Lambda_n[e]) \quad D_T(h \otimes x) = \sum_{j=1}^n T_j h \otimes E_j x$$

$$D_T^2 \equiv 0 \quad \text{so} \quad \text{ran } D_T \subseteq \ker D_T$$

$$\boxed{\sigma_T((T_1, \dots, T_m), \mathcal{H}) = \{\lambda \in \mathbb{C}^m : \text{ran } D_{T-\lambda} \subsetneq \ker D_{T-\lambda}\}}$$

$$m = 3 \quad T = (T_1, T_2, T_3) \quad T_i T_j = T_j T_i$$

$$D_T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ T_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -T_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -T_2 & -T_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_1 & 0 & -T_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_1 & T_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_1 & T_2 & T_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ Q_2 & 0 & 0 & 0 \\ 0 & Q_1 & 0 & 0 \\ 0 & 0 & Q_0 & 0 \end{bmatrix}$$

where  $\text{ran } Q_{j+1} \subseteq \ker Q_j$

$0 \notin \sigma_T((T_1, T_2, T_3), \mathcal{H})$  iff the following complex is exact:

$$0 \rightarrow \mathcal{H} \xrightarrow[\substack{\xrightarrow{Q_2} \\ 1}]{} \bigoplus_1^3 \mathcal{H} \xrightarrow[\substack{\xrightarrow{Q_1} \\ 1}]{} \begin{pmatrix} -T_2 & -T_3 & 0 \\ T_1 & 0 & -T_3 \\ 0 & T_1 & T_2 \end{pmatrix} \bigoplus_1^3 \mathcal{H} \xrightarrow[\substack{\xrightarrow{Q_0} \\ 1}]{} \xrightarrow{(T_1, T_2, T_3)} \mathcal{H} \rightarrow 0$$

$\Leftrightarrow D_T + D_T^*$  is invertible on  $\bigoplus_1^{2^3} \mathcal{H}$

(1) (Koszul, Hilbert) say  $\underline{a} = (a_1, a_2, a_3) \neq \underline{0} \in \mathbb{C}^3$

$$Q_0(\underline{a}) := [a_1, a_2, a_3] : \mathbb{C}^3 \rightarrow \mathbb{C}^1$$

$$\ker Q_0(\underline{a}) = \text{ran } ? = \text{ran } \begin{bmatrix} -a_2 & -a_3 & 0 \\ a_1 & 0 & -a_3 \\ 0 & a_1 & a_2 \end{bmatrix}$$

$$\text{Let } S(a_1, a_2, \dots) = (a_2, a_3, \dots)$$

$$Q_{n+1}(\underline{a}) := \begin{bmatrix} -Q_n(S(\underline{a})) & 0 \\ a_1 I & Q_{n+1}(S(\underline{a})) \end{bmatrix}$$

Then

$$[Q_n(\underline{a})]^* Q_n(\underline{a}) + Q_{n+1}(\underline{a}) [Q_{n+1}(\underline{a})]^* = \|\underline{a}\|_2^2 I$$

$$E_n Q_n + Q_{n+1} E_{n+1} = I$$

$$(2) Ex : \quad Q_0(A) := \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} : \mathbb{C}^4 \rightarrow \mathbb{C}^2 \quad \text{rank } 2$$

(a)  $\ker Q_0(A) = \text{ran } \underbrace{[Q_1(\underline{a}_1) \ Q_2(\underline{a}_2)]}_{Q_1(A)}$

(b) What about  $\ker Q_1(A)$  etc? *Putinar*

(2a) used by *T.* for best estimates in the matrix corona theorem for D.

$\{f_j\}_{j=1}^m \subseteq H^\infty(\mathbb{T}^n) \quad T_j = T_{f_j}$  analytic Toeplitz  
 i.e.  $T_{f_j}(h) = f_j h$

$$(T_1, \dots, T_m) = (T_{f_1}, \dots, T_{f_m}) \text{ onto} \\ \Leftrightarrow \sum_{j=1}^m T_{f_j} T_{f_j}^* \geq \epsilon^2 I, \epsilon > 0$$

always have  $\begin{pmatrix} (-1)^{m+1} T_{f_m} \\ \vdots \\ T_{f_1} \end{pmatrix} 1 - 1$

## Theorem

$$\sigma_T((T_{f_1}, \dots, T_{f_m}), \mathcal{H}) = \overline{\text{ran } F(D^n)}^{\mathbb{C}^m}$$

$$\sigma_T((T_{f_1}, \dots, T_{f_m}), \mathcal{H})$$

$$\stackrel{(\textcolor{blue}{T}_.)}{=} \{ \lambda \in \mathbb{C}^m : \nexists \epsilon > 0, \sum_{j=1}^m (T_{f_j} - \lambda_j)(T_{f_j} - \lambda_j)^* \geq \epsilon^2 I \}$$

$$\stackrel{(*)}{=} \{ \lambda \in \mathbb{C}^m : \inf_{z \in D^n} \sum_{j=1}^m |f_j(z) - \lambda_j|^2 = 0 \}$$

$$= \{(f_1(z), \dots, f_m(z)) : z \in D^n\}^{-\mathbb{C}^m}$$

$$= \overline{\text{ran } F(D^n)}^{\mathbb{C}^m}$$

(\*) Li, Lin, Boo, Varopoulos, others

(for the ball and strictly pseudoconvex domains - Putinar, Wolff, Eschmeier, Andersson-Carlsson, ...) Note: Can replace  $\mathcal{H}$  with  $H^p(T^n)$  for  $1 \leq p < \infty$

## Example

$$m = 3, n = 2$$

Assume  $(T_{f_1}, T_{f_2}, T_{f_3})$  is onto,

$$Q_0(z) = (f_1(z), f_2(z), f_3(z)) \text{ and } T_j = T_{f_j}$$

If

$$Q_0 \underline{h} = (T_1, T_2, T_3) \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \underline{0}, \quad h_i \in H^2(T^2),$$

then we want

$$\underline{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} -T_2 & -T_3 & 0 \\ T_1 & 0 & -T_3 \\ 0 & T_1 & T_2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = Q_1 \underline{k}$$

where  $k_i \in H^2(T^2)$

Similarly,

$$Q_1 \underline{r} = \underline{0} \Rightarrow \underline{r} = Q_2(\underline{m}) \text{ and } Q_2 \text{ always } 1 - 1,$$

$$\text{so } \underline{0} \notin \sigma_T((T_1, T_2, T_3), \mathcal{H})$$

## Basic Idea (Poincare, Dolbeault-Grothendieck)

$$Q_0(z) := F(z) = (f_1(z), f_2(z), f_3(z))$$

$$m = 3, n = 2$$

Define:

$$D_1 = \begin{pmatrix} \bar{\partial}_1 \\ \bar{\partial}_2 \end{pmatrix}, D_2 = (-\bar{\partial}_2, \bar{\partial}_1)$$

Then  $0 = Q_0 Q_1 = Q_1 Q_2$ ,  $D_2 D_1 = 0$ , and  $D_i \leftrightarrow Q_j$

## Example

$$D_1 Q_2 \underline{\varphi} = \begin{pmatrix} \bar{\partial}_1(Q_2 \underline{\varphi}) \\ \bar{\partial}_2(Q_2 \underline{\varphi}) \end{pmatrix}$$

$$Q_2 D_1 \varphi = \begin{pmatrix} Q_2 \bar{\partial}_1 \varphi \\ Q_2 \bar{\partial}_2 \varphi \end{pmatrix}$$

So

$$D_1 \leftrightarrow Q_2$$

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Then For  $E_j(z) = \frac{Q_j(z)^*}{F(z)F(z)^*}$ ,

$$Q_0 E_0 = I \quad \mathcal{K}_1 D_1 = P_{H^2(T^2)}^\perp$$

$$E_0 Q_0 + Q_1 E_1 = I \quad D_1 \mathcal{K}_1 + \mathcal{K}_2 D_2 = I$$

$$E_1 Q_1 + Q_2 E_2 = I \quad D_2 \mathcal{K}_2 = I$$

$$E_2 Q_2 = I$$

$\mathcal{K}_i$  are integral operators on smooth functions on  $\overline{D}^2$

Say  $T_{Q_0} \underline{h} = 0$  so  $Q_0(z) \underline{h}(z) \equiv 0$  and  $E_0 Q_0 + Q_1 E_1 = I \Rightarrow$

$$Q_1(z)[E_1(z) \underline{h}(z)] = \underline{h}(z)$$

Modify:  $E_1(z) \underline{h}(z) - Q_2(z) \textcolor{red}{X}(z)$  *analytic* note:  $Q_1 Q_2 = 0$

$$\underline{k} = E_1 \underline{h} - Q_2 \textcolor{red}{K}_1 \textcolor{blue}{E}_2 D_1 E_1 \underline{h}$$

$$P_{H^2}^\perp \underline{k} = \textcolor{red}{K}_1 \textcolor{red}{D}_1 [E_1 \underline{h} - Q_2 K_1 E_2 D_1 E_1 \underline{h}] = \dots = 0$$

General case:

$$Q_0(z) = (f_1(z), \dots), z \in D^n, I \geq T_{Q_0} T_{Q_0}^* \geq \epsilon^2 I, \epsilon > 0$$

$$T_{Q_j}(\underline{a}) = \underline{0} \quad \underline{a} \in \bigoplus_1^\infty H^2(T^n)$$

$$\underline{k} = E_{j+1} \underline{a} + \sum_{p=1}^n (-1)^{p+1} Q_{j+2} K_1 \cdots Q_{p+j+1} K_p E_{p+j+1} D_p \cdots E_{j+2} D_1 E_{j+1} \underline{a}$$

$$\in \bigoplus_1^\infty H^2(T^n)$$

n=1

$$K_1 = \Lambda_1 \text{ and } D_1 = \bar{\partial}_1$$

$$\text{and } K_1 D_1 = P_{H^2(T)}^\perp$$

$$D_1 K_1 = I$$

$$\text{e.g. } \Lambda_1(\varphi)(e^{it}, e^{is}) = -\frac{1}{\pi} \int_D \frac{\varphi(u, e^{is})}{u - e^{it}} dm(u)$$

n=2

$$D_1 = \begin{pmatrix} \bar{\partial}_1 \\ \bar{\partial}_2 \end{pmatrix}, D_2 = (-\bar{\partial}_2, \bar{\partial}_1)$$

$$K_1 = (P_2 \Lambda_1, \Lambda_2) \text{ and } K_2 = \begin{pmatrix} \Lambda_2 \\ 0 \end{pmatrix}$$

where  $P_2$  projects  $L^2(T) \otimes L^2(T)$  onto  $L^2(T) \otimes H^2(T)$   
and  $\Lambda_i$  is the Cauchy transform on the  $i$ th variable

## General n

$$K_1(n) = (P_n P_{n-1} \cdots P_2 \Lambda_1, P_n \cdots P_3 \Lambda_2, \dots, \Lambda_n)$$

$$K_{j+1}(n) = \begin{bmatrix} -K_j^+(n-1) & 0 \\ 0 & K_{j+1}^+(n-1) \end{bmatrix}$$

$$D_{j+1}(n) = \begin{bmatrix} -D_j^+(n-1) & \bar{\partial}_1 \otimes I \\ 0 & D_{j+1}^+(n-1) \end{bmatrix}$$

Key fact in estimates on  $D^n$ :

$$F = (f_1, \dots), 1 \geq F(z)F(z)^* \geq \epsilon^2 > 0, \forall z \in D^n$$

$$H \in H^1(T^n), I_1 \dot{\cup} I_2 \dot{\cup} \cdots \dot{\cup} I_j \dot{\cup} J = \{1, 2, \dots, n\}$$

$$F_{I_1} = \partial_{i_1} \cdots \partial_{i_p} F, \text{ where } I_1 = \{i_1, \dots, i_p\}$$

$$\int_D \cdots \int_D \|F_1\|_2 \dots \|F_n\|_2 \|F_{I_1}\|_2 \cdots \|F_{I_j}\|_2 |H_J| dL_{I_1} \cdots dL_{I_j} dL_J \leq C_0 \|H\|_1$$

$$dL_{I_1} = \prod_{j=1}^p \ln \left( \frac{1}{1 - |z_{i_j}|} \right) dm(z_{i_j})$$

Chang  $n = 2$ ; Terwilliger-Lacey-Ferguson

$$H = \sum f_j g_j \quad \|H\|_1 \sim \sum \|f_j\|_2 \|g_j\|_2$$

## Application I

$$F = (f_1, f_2, \dots), f_j \in H^\infty(D)$$

First, recall

**Corona Theorem for  $H^\infty(D)$**

(Carleson, Rosenblum, Tolokonnikov, Uchiyama), Wolff

$$0 < \epsilon^2 \leq F(z)F(z)^* \leq 1, z \in D, \text{ then}$$

$\exists \{u_j\}_{j=1}^\infty \subseteq H^\infty(D)$  such that

$$(a) \sum_{j=1}^{\infty} f_j u_j \equiv 1$$

$$(b) \sup_{z \in D} \sum_{j=1}^{\infty} |u_j(z)|^2 < \infty$$

$H^\infty(D)$  is the multiplier algebra for  $H^2(T)$ .

(A)  $H^2(T)$  Corona Theorem

$$0 < \epsilon^2 \leq F(z)F(z)^* \leq 1, z \in D,$$

then

$$\exists \delta > 0 \text{ such that } I \geq T_F T_F^* \geq \delta^2 I$$

(B) Operator Corona Theorem

$$I \geq T_F T_F^* \geq \delta^2 I, \text{ then } \exists \{u_j\}_{j=1}^{\infty} \subseteq H^{\infty}(D)$$

such that

$$U^T = (u_1, u_2, \dots), \text{ then (a) } T_F T_U = I$$
$$\text{(b) } T_U \text{ bounded}$$

(Schubert, Rosenblum, Nagy-Foias, Arveson, . . . )

(B) CLT [reproducing kernel has 1-positive square, complete NP]

$$1 - \frac{1}{k_w(z)} = \sum_{n=1}^{\infty} a_n(z) \overline{a_n(w)} \quad a_n \in \mathcal{H}$$

[Ball-T.- Vinnikov][McCullough,Quiggen][Agler]

Ex:  $\frac{1}{1-z\bar{w}}$  in  $H^2(T)$    Non-Ex:  $\frac{1}{(1-z\bar{w})^2}$  in  $B^2(D)$

$$\frac{1}{z\bar{w}} \text{ in } \frac{1}{1-z\bar{w}} \text{ in } \mathcal{D}^2(D) \qquad \prod_{j=1}^n \frac{1}{(1-z_j\bar{w}_j)} \text{ in } H^2(T^n) \quad n > 1$$

$$\frac{1}{1 - \sum_{i=1}^n z_i \bar{w}_i} \text{ in } \mathcal{F}^n \qquad \frac{1}{(1 - \sum_{i=1}^n z_i \bar{w}_i)^n} \text{ in } H^2(B^n) \quad n > 1$$

( $\mathcal{F}^n$  symmetric Fock space or Drury-Arveson space)

Aside:  $\mathcal{A} = \{\text{analytic Toeplitz operators on } H^2(T)\}$

$$A \in \mathcal{A} \text{ and } AA^* \geq \epsilon^2 I$$

**Operator Corona Theorem**  $\Rightarrow \exists B \in \mathcal{A} \text{ s.t. } AB = I \text{ and } \|B\| \leq \frac{1}{\epsilon}$

Of course, for any subalgebra,  $\mathcal{A}$ , of  $B(\mathcal{H})$

$$A \in \mathcal{A}, AA^* \geq \epsilon^2 I$$

$$\Rightarrow \exists X \in B(\mathcal{H}) \text{ s.t. } AX = I \text{ and } \|X\| \leq \frac{1}{\epsilon}$$

but  $X$  might not belong to  $\mathcal{A}$

**Problem 1** Classify, unital, “closed” algebras  $\mathcal{A}$  in  $B(H)$   
s.t., whenever  $B, A \in \mathcal{A} \otimes B(\mathcal{H})$  with  $AA^* \geq BB^*$

$$\exists C \in \mathcal{A} \otimes B(H)$$

$$\text{with } \|C\| \leq 1$$

$$\text{and } B = AC$$

Replace  $H^2(T)$  by  $\mathcal{D}_\alpha^2(D)$

$H^\infty(D)$  by  $\mathcal{M}(\mathcal{D}_\alpha^2(D))$

$\forall z \in D, F(z)F(z)^* \leq 1$  by  $\|(M_{f_1}, M_{f_2}, \dots)^T\| < \infty$

$\sup_{z \in D} \sum_{j=1}^{\infty} |u_j(z)|^2$  by  $\|(M_{u_1}, M_{u_2}, \dots)^T\| < \infty$

(B)(Operator corona Thm) follows since repro. ker. has 1-positive square

Corona theorem for  $\mathcal{M}(\mathcal{D}_\alpha^2(D))$

(finite case, Tolokonnikov;

$\infty$  case and  $\alpha = 1$  T.;

$\infty, 0 < \alpha < 1$  Kidane)

$n$ -variable version Besov spaces, in particular, symmetric Fock space,  $\mathcal{F}^n$  !

(Arocena-Sawyer-Wick)

## Application II

$F = (f_1, f_2, \dots)$ ,  $f_i \in A$  closed, unital subalgebra  $H^\infty(D)$

(a) ([Scheinberg](#))  $\exists$  closed unital subalgebras,  $A$ , of  $H^\infty(D)$  for which the corona theorem **fails**

(b)  $A = H_K^\infty(D)$  where  $K \subset \{1, 2, \dots\}$

$f \in H_K^\infty(D) \Rightarrow f^{(j)}(0) = 0, \forall j \in K$

and  $H_K^\infty(D)$  is an **algebra**

( $K = \{1\}$ , [Davidson-Paulsen-Ragupathi-Singh](#) interpolation

$K = \{1\}$  and **finite corona thm**, [Mortini-Sasane-Wick](#)

**General  $K$**  and **infinite corona thm**, [Ryle](#))

## Example

$$\mathbb{N} - \textcolor{red}{K} = \{0, 6, 8, 2n : n \geq 6\}$$

$$f \in H_{\textcolor{red}{K}}^\infty(D), f(z) = f_0 + f_6 z^6 + f_8 z^8 + \sum_{n=6}^{\infty} f_{2n} z^{2n}$$

Consider

$$F(w) = f_0 + f_6 w^3 + f_8 w^8 + w^6 \textcolor{red}{G}(w), \textcolor{red}{G} \in H^\infty(D)$$

## Application III

$H^2(T^n)$  or  $H^2(B^n)$ )

(A)  $H^2(T^n)$ -corona theorem Li, Lin, Boo, ...

Thm(T., Wick) corona theorem for  $H^\infty(D^n)$  holds

$$\Leftrightarrow \forall 0 < \epsilon^2 \leq F(z)F(z)^* \leq 1 \text{ on } D^n,$$
$$\forall \tilde{H} \in H^2(T^n),$$

$H$  nonvanishing on  $D^n$  and  $\frac{1}{\tilde{H}} \in L^\infty(T^n)$

$H^2(|\tilde{H}|^2 d\sigma_1 \dots d\sigma_n)$  - corona theorem holds

Note: (a) (Agler-McCarthy,Amar)

(b)  $H^2(|\tilde{H}|^2 d\sigma_1 \dots d\sigma_n)$  - corona theorem holds if  $\tilde{H}$  is outer

(c) same result on  $B^n$

(d) suffices  $1 \in \text{ran}(T_F^{|\tilde{H}|^2 d\sigma_1 \dots d\sigma_n})$

i.e.,  $F U = 1$ ,  $U \in \bigoplus_1^\infty H^2(T^n)$  and  $\|U \tilde{H}\|_2 < \infty$

Key facts:

$$(1) \quad F a \equiv 1, \quad a \in \bigoplus_1^{\infty} H^2(T^n)$$

then since  $\ker T_F = \text{ran } T_{Q_1}$

$$a = T_F^*(T_F T_F^*)^{-1}(1) - T_{Q_1} b, \quad b \in \bigoplus_1^{\infty} H^2(T^n)$$

(2) minimax

(3) Rudin-Polydisk

Løw, Alexandrov-Ball

## Problem 2

$\tilde{H} \in H^2(T^2)$  and  $|\tilde{H}| \geq 1$  a.e. on  $T^2$   
 $H$  nonvanishing on  $D^2$

Does  $\exists$  outer  $K \in H^2(T^2)$  such that  $|\tilde{H}| \leq |K|$  on  $T^2$ ?

Note:  $H = (\frac{H}{K})K$ ,  $\frac{H}{K} \in H^\infty(D^2)$ , so there is a counterexample if  $H$  has 0's in  $D^2$ !

## Problem 3

$F = [f_{ij}]$ ,  $f_{ij} \in H^\infty(D)$

Suppose that

$M_F \in B(\bigoplus_1^\infty A^2(D))$  and  $M_F M_F^* \geq \epsilon^2 I$ ,  $\epsilon > 0$

Is  $T_F T_F^*$  invertible?

Here  $A^2(D) = \{f \in Hol(D) : \int_D |f(z)|^2 dm_2(z) < \infty\}$

(Note:  $\epsilon^2 I \leq F(z)F(z)^* \leq C_0 I \quad \forall z \in D$  Treil counterexample)

$$F(z) = [f_{ij}(z)] \quad f_{ij} \in H^\infty(D)$$

Assume that

$$\epsilon^2 I \leq F(z)F(z)^* \leq I \quad \forall z \in D$$

(Nagy) Does there exist  $G(z) = [g_{ij}(z)] \quad g_{ij} \in H^\infty(D)$

s.t.  $F(z)G(z) \equiv I \quad z \in D$

and  $\sup_{z \in D} \|G(z)\|_{B(l^2)} < \infty?$

Treil -No!

Thus, if the answer to Problem 3 is “yes”,

then the corona theorem holds for  $H^\infty(D^2)$

If the answer to Problem 3 is “no”,

then there is a stronger counterexample than Treil's,

since in Problem 3, we assume that  $F$  also satisfies

$$M_F \in B\left(\bigoplus_1^\infty A^2(D)\right)$$

and ask the Nagy question.