Bounded analytic functions on the infinite polydisc

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Contents

The talk is a report on some recent work on Dirichlet series and bounded analytic functions on finite and infinite polydiscs.

- The first part is about the Bohnenblust–Hille inequality for homogeneous polynomials; it is based on joint work with A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes (arXiv:0904.3540v1, 2009)
- The second part is about boundary limit functions of elements in Hardy spaces of Dirichlet series; it is based on joint work with E. Saksman (Bull. London Math. Soc. **41** (2009).

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- Is it possible to have a similar inequality

$$\left(\sum_{|\alpha|=m} |a_\alpha|^p\right)^{\frac{1}{p}} \leq C\|P\|_\infty$$

for some $p < 2$ with C depending on m but *not* on n?

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YES, and $2m/(m+1)$ is the smallest possible p.

It is of basic interest to know the asymptotic behavior of C when $p = 2m/(m+1)$ and $m \to \infty$.

A multilinear inequality

In 1930, Littlewood proved that for every bilinear form $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ we have

$$\left(\sum_{i,j} |B(e^i, e^j)|^{4/3} \right)^{3/4} \leq \sqrt{2} \sup_{z,w \in \mathbb{D}^n} |B(z, w)|.$$

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This was extended to m-linear forms by Bohnenblust and Hille in 1931:

$$\Big(\sum_{i_1,\dots,i_m} |B(e^{i_1}, \dots, e^{i_m})|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \leq \sqrt{2}^{m-1} \sup_{z^i \in \mathbb{D}^n} |B(z^1, \dots, z^m)|.$$

The exponent $2m/(m+1)$ is best possible.

The Bohnenblust–Hille inequality

Our result is that also the polynomial Bohnenblust–Hille inequality is hypercontractive:

Theorem (Defant, Frerick, Ortega-Cerdà, Ounaïes, Seip 2009)

Let m and n be positive integers larger than 1. Then we have

$$\Big(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le e\sqrt{m}(\sqrt{2})^{m-1} \sup_{z\in\mathbb{D}^n}\Big|\sum_{|\alpha|=m} a_\alpha z^\alpha\Big|$$

for every m-homogeneous polynomial $\sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$.

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for every m-homogeneous polynomial $\sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$.

The novelty here is the **hypercontractivity**, i.e., the constant grows exponentially with m; known since the work of Bohnenblust–Hille that the inequality holds with constant $m^{m/2}$, modulo a factor of exponential growth.

Polarization

There is an obvious relationship between the multilinear and polynomial inequalities, which goes via a one-to-one correspondence between symmetric multilinear forms and homogeneous polynomials.

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Definition

We say that the m-linear form B is symmetric if $B(e^{i_1}, ..., e^{i_m}) = B(e^{i_{\sigma(1)}}, ..., e^{i_{\sigma(m)}})$ for every index set $(i_1, ..., i_m)$ and every permutation σ of the set $\{1, ..., m\}$.

Polarization—continued

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$$P(z) = \sum_{i_1 \le \cdots \le i_m} c(i_1, \ldots, i_m) z_{i_1} \cdots z_{i_m},$$

and let B be the symmetric m-multilinear form such that $B(e^{i_1}, \cdots, e^{i_m}) = c(i_1, \ldots, i_m)/|i|$ when $i_1 \le \cdots \le i_m$ and $|i|$ is the number of different indices that can be obtained from the index $i = (i_1, \ldots, i_m)$ by permutation.

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Harris's lemma

Lemma (Harris 1975)

We have

$$\sup_{z^i \in \mathbb{D}^n} |B(z^1, \ldots, z^m)| \leq \frac{m^m}{m!} \|P\|_\infty.$$

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Since the number of coefficients of B obtained from one coefficient of P is bounded by $m!$, a direct application of Harris's lemma and the multilinear Bohnenblust–Hille inequality gives

$$\left(\frac{1}{m!}\right)^{(m-1)/2m} \Big(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \leq \frac{m^m}{m!} \|P\|_\infty ;$$

we obtain then the afore-mentioned constant $m^{m/2}$, modulo an exponential factor.

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Therefore, to make the required improvement, one needs a refinement of the argument via multilinear forms.

Two lemmas

Lemma (Blei 1979)

For all sequences $(c_i)_i$ where $i = (i_1, \dots, i_m)$ and $i_k = 1, \dots, n$, we have

$$\Big(\sum_{i_1,\dots,i_m=1}^{n} |c_i|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \le \prod_{1\le k\le m} \Big[\sum_{i_k=1}^{n} \Big(\sum_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_m} |c_i|^2 \Big)^{\frac{1}{2}} \Big]^{\frac{1}{m}}.$$

Lemma (Bayart 2002)

For any homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$:

$$\Big(\sum_{|\alpha|=m} |a_\alpha|^2 \Big)^{\frac{1}{2}} \le (\sqrt{2})^m \Big\| \sum_{|\alpha|=m} a_\alpha z^\alpha \Big\|_{L^1(\mathbb{T}^n)}.$$

The proof 1/3

We write again the polynomial P as

$$P(z) = \sum_{i_1 \leq \cdots \leq i_m} c(i_1, \ldots, i_m) z_{i_1} \cdots z_{i_m}.$$

We have

$$\sum_{i_1 \leq \cdots \leq i_m} |c(i_1, \ldots, i_m)|^{2m/(m+1)} \leq \sum_{i_1, \ldots, i_m} \left(\frac{|c(i_1, \ldots, i_m)|}{|i|^{1/2}} \right)^{2m/(m+1)}$$

By Blei's lemma, the last sum is bounded by

$$\prod_{k=1}^{m} \Big[\sum_{i_k=1}^{n} \Big(\sum_{i^k} \frac{|c(i_1, \ldots, i_m)|^2}{|i|} \Big)^{1/2} \Big]^{1/m} \leq \sqrt{m} \prod_{k=1}^{m} \Big[\sum_{i_k=1}^{n} \Big(\sum_{i^k} |i^k| \frac{|c(i_1, \ldots, i_m)|^2}{|i|^2} \Big)^{1/2} \Big]^{1/m}.$$

The proof 2/3

We now freeze the variable i_k and group the terms to make a polynomial again:

$$\Big(\sum_{i^k} |i^k| \frac{|c(i_1,\ldots,i_m)|^2}{|i|^2}\Big)^{1/2} = \Big(\sum_{i^k} |i^k| |B(e^{i_1},\ldots,e^{i_m})|^2\Big)^{1/2} = \|P_k\|_2.$$

where $P_k(z)$ is the polynomial $P_k(z) = B(z,\ldots,z,e^{i_k},z,\ldots,z)$. Now we use Bayart's estimate and get

$$\Big(\sum_{i^k} |i^k| \frac{|c(i_1,\ldots,i_m)|^2}{|i|^2}\Big)^{1/2} \leq \sqrt{2}^{m-1} \int_{\mathbb{T}^n} |B(z,\ldots,z,e^{i_k},z,\ldots,z)|.$$

The proof 3/3

We replace e^{i_k} by λe^{i_k} with $|\lambda| = 1$. We take $\tau_k(z) = \sum \lambda_k(z) e^{i_k}$ in such a way that

$$\sum_{i_k=1}^{n} \Big(\sum_{i^k} |i^k| \frac{|c(i_1, \ldots, i_m)|}{|i|^2} \Big)^{1/2} \leq \sqrt{2}^{m-1} \int_{\mathbb{T}^n} B(z, \ldots, \tau_k(z), \ldots, z)$$
$$\leq e^m \sqrt{2}^{m-1} \|P\|_\infty,$$

where in the last step we used Harris's lemma. Finally,

$$\sum_{i_1 \leq \cdots \leq i_m} |c(i_1, \ldots, i_m)|^{2m/(m+1)} \leq e^m \sqrt{2}^{m-1} \sqrt{m} \|P\|_\infty.$$

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where in the last step we used Harris's lemma. Finally,

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(The factor e^m can be reduced to e by use of a refined version of Harris's lemma.)

Consequences of the hypercontractive BH inequality

Our improvement of the polynomial Bohnenblust–Hille inequality may seem marginal, but it has several interesting consequences: It leads to precise asymptotic results regarding certain Sidon sets, Bohr radii, and absolute convergence of Dirichlet series.

Sidon sets

Definition

If G is an Abelian compact group and Γ its dual group, a subset of the characters $S \subset \Gamma$ is called a Sidon set if

$$\sum_{\gamma \in S} |a_\gamma| \leq C \| \sum_{\gamma \in S} a_\gamma \gamma \|_\infty$$

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We estimate the Sidon constant for homogeneous polynomials:

Definition

$S(m, n)$ is the smallest constant C such that the inequality $\sum_{|\alpha|=m} |a_\alpha| \leq C \|P\|_\infty$ holds for every m-homogeneous polynomial in n complex variables $P = \sum_{|\alpha|=m} a_\alpha z^\alpha$.

The Sidon constant for homogeneous polynomials

Since the number of different monomials of degree m is $\binom{n+m-1}{m}$, Hölder's inequality gives:

Corollary

Let m and n be positive integers larger than 1. Then

$$S(m,n) \leq e\sqrt{m}(\sqrt{2})^{m-1}\binom{n+m-1}{m}^{\frac{m-1}{2m}}.$$

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(We also have the trivial estimate

$$S(m,n) \le \sqrt{\binom{n+m-1}{m}},$$

so the corollary is of interest only when $\log n \gg m$.)

The n-dimensional Bohr radius

Definition

The n-dimensional Bohr radius K_n is the largest $r > 0$ such that all polynomials $\sum_\alpha c_\alpha z^\alpha$ satisfy

$$\sup_{z \in r\mathbb{D}^n} \sum_\alpha |c_\alpha z^\alpha| \le \sup_{z \in \mathbb{D}^n} \left| \sum_\alpha c_\alpha z^\alpha \right|.$$

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When $n > 1$, the precise value of K_n is unknown.

Problem

Determine the asymptotic behavior of K_n when $n \to \infty$.

Footnote on F. Wiener

The initial ‘F’ is not a misprint: F. Wiener is the mathematician Friedrich Wilhelm Wiener, born in 1884, and probably a casualty of World War One. See Boas and Khavinson’s biography `http://arxiv.org/PS_cache/math/pdf/9901/9901035v1.pdf`.

Asymptotic behavior of K_n

The problem was studied by Boas and Khavinson in 1997. They showed that

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Theorem (DFOOS 2009)

The n-dimensional Bohr radius satisfies

$$c\sqrt{\frac{\log n}{n}} \leq K_n \leq 2\sqrt{\frac{\log n}{n}}.$$

Proof of theorem on the Bohr radius

We use a well known lemma of F. Wiener.

Lemma

Let P be a polynomial in n variables and $P = \sum_{m \geq 0} P_m$ its expansion in homogeneous polynomials. If $\|P\|_\infty \leq 1$, then $\|P_m\|_\infty \leq 1 - |P_0|^2$ for every $m > 0$.

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We assume that $\sup_{\mathbb{D}^n}\left|\sum a_\alpha z^\alpha\right| \leq 1$. Observe that for all z in $r\mathbb{D}^n$,

$$\sum |a_\alpha z^\alpha| \leq |a_0| + \sum_{m>1} r^m \sum_{|\alpha|=m} |a_\alpha|.$$

End of proof

If we take into account the estimates

$$\frac{(\log n)^m}{n} \leq m! \qquad \text{and} \qquad \binom{n+m-1}{m} \leq e^m \big(1+\frac{n}{m}\big)^m,$$

then we use the Sidon estimate and F. Wiener lemma:

$$\sum_{m>1} r^m \sum_{|\alpha|=m} |a_\alpha| \leq \sum_{m>1} r^m e\sqrt{m}(2\sqrt{e})^m \Big(\frac{n}{\log n}\Big)^{m/2} (1-|a_0|^2).$$

Choosing $r \leq \varepsilon\sqrt{\frac{\log n}{n}}$ with ε small enough, we obtain

$$\sum |a_\alpha z^\alpha| \leq |a_0| + (1-|a_0|^2)/2 \leq 1$$

whenever $|a_0| \leq 1$.

Remark: What is the “Bohr subset” of $\mathbb{D}^n$?

A more difficult problem would be to find the “Bohr subset” of $\mathbb{D}^n$, i.e., the set of points z in $\mathbb{D}^n$ for which

$$\sum_{\alpha} |c_\alpha z^\alpha| \leq \sup_{z \in \mathbb{D}^n} \Big| \sum_{\alpha} c_\alpha z^\alpha \Big|$$

holds for all polynomials $\sum_\alpha c_\alpha z^\alpha$.

The BH inequality and Dirichlet Series

Recall that an ordinary Dirichlet series is a series of the form $\sum_{n\geq 1} a_n n^{-s}$, where the exponentials n^{-s} are positive for positive arguments s. The original work of Bohnenblust and Hille (1931) was motivated by a problem of Bohr from 1913 on the convergence of such series.

Convergence of Dirichlet series

In general, a Dirichlet series has several half-planes of convergence, as shown in the picture:

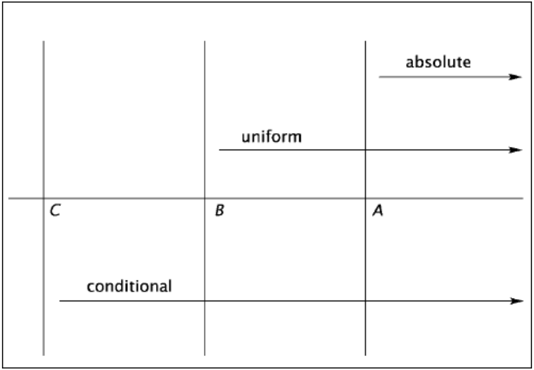

Figure: Convergence regions for Dirichlet series

Bohr's problem on absolute convergence

It is plain that $0 \le A - C \le 1$, and if $a_n = e^{in^\alpha}$ with $0 \le \alpha \le 1$, then $C = 1 - \alpha$ and $A = 1$.

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The most interesting quantity is the difference $A - U$. Bohr proved that it does not exceed $1/2$, but he was unable to exhibit even one example such that $A - U > 0$.

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Alternate viewpoint: Bohr proved that the abscissa of uniform convergence is the same as the abscissa of boundedness and regularity, i.e. the infimum of those σ_0 such that the function represented by the Dirichlet series is analytic and bounded in $\Re s = \sigma > \sigma_0$. Thus we may instead look at A for bounded analytic functions represented by Dirichlet series.

The Bohnenblust–Hille theorem on absolute convergence

Definition

The space $\mathscr{H}^\infty$ consists of those bounded analytic functions f in $\mathbb{C}_+ = \{s = \sigma + \mathrm{i}\, t : \ \sigma > 0\}$ such that f can be represented by an ordinary Dirichlet series $\sum_{n=1}^\infty a_n n^{-s}$ in some half-plane.

The Bohnenblust–Hille theorem can be rephrased as:

Theorem

The infimum of those $c > 0$ such that

$$\sum |a_n| n^{-c} < +\infty$$

for every $\sum_{n=1}^\infty a_n n^{-s}$ in $\mathscr{H}^\infty$ equals $1/2$.

A refined version of the Bohnenblust–Hille theorem

Theorem

The supremum of the set of real numbers c such that

$$\sum_{n=1}^{\infty} |a_n|\, n^{-\frac{1}{2}} \exp\Big\{c\sqrt{\log n \log\log n}\Big\} < \infty$$

for every $\sum_{n=1}^{\infty} a_n n^{-s}$ *in* $\mathscr{H}^\infty$ *equals* $1/\sqrt{2}$.

This is a refinement of a theorem of R. Balasubramanian, B. Calado, and H. Queffélec (2006). (Without the hypercontractive BH inequality, one does not catch the precise bound for the constant c.)

Bohr's insight

Let $f(s) = \sum_{n\geq 1} a_n n^{-s}$ be a Dirichlet series. We factor each integer n into a product of prime numbers $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and set $z = (p_1^{-s}, p_2^{-s}, \ldots)$. Then

$$f(s) = \sum_{n=1}^{\infty} a_n (p_1^{-s})^{\alpha_1} \cdots (p_r^{-s})^{\alpha_r} = \sum a_n z_1^{\alpha_1} \cdots z_r^{\alpha_r}.$$

Bohr's correspondence is not just formal. The space $\mathscr{H}^\infty$ is isometric to the space $H^\infty(\mathbb{T}^\infty) := L^\infty(\mathbb{T}^\infty) \cap H^2(\mathbb{T}^\infty)$ (or $H^\infty(\mathbb{D}^\infty)$ which can be defined as the set of bounded analytic functions on $\mathbb{D}^\infty \cap c_0$), thanks to a classical result of Kronecker on diophantine approximation.

Bohr's insight—continued

Bohr's correspondence is an indispensable tool for proving nontrivial results about Dirichlet series.

Bohr's insight—continued

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The proof of the refined version of the theorem of Balasubramanian–Calado–Queffélec goes via the following beautiful result about Dirichlet polynomials; it combines the Bohnenblust–Hille inequality with probabilistic methods and methods from analytic number theory.

Estimates on coefficients of Dirichlet polynomials

For a Dirichlet polynomial

$$Q(s) = \sum_{n=1}^{N} a_n n^{-s},$$

we set $\|Q\|_\infty = \sup_{t \in \mathbb{R}} |Q(it)|$ and $|||Q|||_1 = \sum_{n=1}^{N} |a_n|$. Then $S(N)$ is the smallest constant C such that the inequality $|||Q|||_1 \leq C\|Q\|_\infty$ holds for every Q.

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Theorem (Konyagin–Queffélec 2001, de la Bretèche 2008, DFOOS 2009)

We have

$$S(N) = \sqrt{N} \exp\Big\{ \big(-\frac{1}{\sqrt{2}} + o(1)\big) \sqrt{\log N \operatorname{loglog} N} \Big\}$$

when $N \to \infty$.

Historical account of the estimate for $S(N)$

The inequality

$$S(N) \geq \sqrt{N}\exp\Big\{\big(-\frac{1}{\sqrt{2}} + o(1)\big)\sqrt{\log N \log\log N}\Big\}$$

was established by R. de la Bretèche, who also showed that

$$S(N) \leq \sqrt{N}\exp\Big\{\big(-\frac{1}{2\sqrt{2}} + o(1)\big)\sqrt{\log N \log\log N}\Big\}$$

follows from an ingenious method developed by Konyagin and Queffélec. The same argument, using the hypercontractive BH inequality at a certain point, gives the sharp result.

More on the function theory of $\mathscr{H}^\infty$

So far, our discussion has centered around the Bohnenblust–Hille inequality. In the remaining part of the talk, we will look at properties of boundary limit functions of elements in $\mathscr{H}^\infty$ and the related spaces $\mathscr{H}^p$. Now we do not use the Bohnenblust–Hille inequality, but the Bohr correspondence is still of basic importance.

Carlson's theorem

Suppose f is in $\mathscr{H}^\infty$. By a direct computation, we get what is known as Carlson's theorem[1]:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |f(\sigma+it)|^2\,dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}.$$

[1] Named after the Swedish mathematician Fritz Carlson (1888-1952).

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Thus

$$\sum_{n=1}^{\infty} |a_n|^2 \leq \|f\|_\infty^2 < \infty.$$

Note that Cauchy-Schwarz gives $A \leq 1/2$, and thus we have a simple proof of Bohr's inequality $A - U \leq 1/2$.

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Ergodicity

We may think of the line $t \mapsto \sigma_0 + it$ as a quasi-periodic motion, or, more precisely, for each prime p, we associate a periodic motion on the circle of radius $p^{-\sigma_0}$:

$$t \mapsto p^{-it} p^{-\sigma_0}.$$

Thus we think of vertical translations as a motion on a system of circles, or, if you like, on an infinite-dimensional torus.

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Thus we think of vertical translations as a motion on a system of circles, or, if you like, on an infinite-dimensional torus. We use the product measure of the normalized arc length measures for the circles. Then the flow is ergodic (by the afore-mentioned approximation theorem of Kronecker), and we may use the Birkhoff–Khinchin ergodic theorem.

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Thus we think of vertical translations as a motion on a system of circles, or, if you like, on an infinite-dimensional torus. We use the product measure of the normalized arc length measures for the circles. Then the flow is ergodic (by the afore-mentioned approximation theorem of Kronecker), and we may use the Birkhoff–Khinchin ergodic theorem. In particular, this gives us a fancier way of proving Carlson's theorem:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |f(\sigma + it)|^2 \, dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}.$$

Carlson's theorem on the boundary?

An interesting question raised by Hedenmalm (2003) is whether Carlson's theorem extends to the imaginary axis.

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To place it in context, I will mention a few related matters before we answer that question.

The Dirichlet–Hardy space $\mathscr{H}^2$

Definition

$\mathscr{H}^2$ consists of all Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ for which

$$\|f\|_2^2 = \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

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$$\|f\|_2^2 = \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

Since by Cauchy-Schwarz

$$|f(s)|^2 \le \|f\|_2^2 \sum_{n=1}^{\infty} n^{-2\sigma},$$

an f in $\mathscr{H}^2$ is analytic in $\mathbb{C}^+_{1/2} = \{s = \sigma + it : \ \sigma > 1/2\}$. In particular, $\mathscr{H}^\infty \subset \mathscr{H}^2$ and $\|f\|_2 \le \|f\|_\infty$. $\mathscr{H}^2$ is, via the Bohr correspondence, the restriction of $H^2(\mathbb{D}^\infty)$ to $\mathbb{C}^+_{1/2}$.

Theorem on multipliers

A multiplier m is a holomorphic function in $\mathbb{C}^+_{1/2}$ such that mf is in $\mathscr{H}^2$ whenever f is in $\mathscr{H}^2$. We denote the set of multipliers by $\mathscr{M}$. Every multiplier m defines a bounded operator on $\mathscr{H}^2$; the corresponding operator norm is denoted by $\|m\|_{\mathscr{M}}$.

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Theorem (Hedenmalm–Lindqvist–Seip 97)

$$\mathscr{M} = \mathscr{H}^\infty \textit{ and } \|m\|_{\mathscr{M}} = \sup_{\sigma>0} |m(\sigma + it)|.$$

The proof of this theorem makes of course essential use of the Bohr correspondence.

Reproducing kernel of $\mathscr{H}^2$

The reproducing kernel of $\mathscr{H}^2$ is

$$K_w(s) = \zeta(s + \overline{w}),$$

where ζ is the Riemann zeta-function defined in $\sigma > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Thus, for f in $\mathscr{H}^2$ and $\sigma > 1/2$, we have

$$f(s) = \langle f, \zeta(\cdot + \overline{s}) \rangle.$$

A pointwise estimate

Since $\zeta(s)$ has a simple pole of residue 1 at 1, we have

$$\zeta(s+\overline{s}) \simeq \frac{1}{2\sigma - 1} + 1,$$

so that

$$|f(s)| \lesssim \frac{\|f\|_2}{\sqrt{\sigma - 1/2}}$$

as $\sigma \to 1/2$. This is the same estimate that governs the growth of functions in $H^2(\mathbb{C}^+_{1/2})$.

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Is there a more precise link to the latter space?

A basic inequality

The following estimate is a mean value inequality that appears in analytic number theory:

$$\int_y^{y+T} |f(1/2+it)|^2 dt \leq CT\|f\|_2^2.$$

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So f is “locally” in $H^2(\mathbb{C}^+_{1/2})$, or, if you like, $f(s)/s$ belongs to $H^2(\mathbb{C}^+_{1/2})$.

Extension of “Carlson’s theorem”?

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For f in $\mathscr{H}^2$, we get

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |f(1/2+it)|^2 \, dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-1}$$

by first proving it for Dirichlet polynomials (trivial) and then extending it to general f using the basic inequality.

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by first proving it for Dirichlet polynomials (trivial) and then extending it to general f using the basic inequality.

However, for $\mathscr{H}^\infty$, Carlson’s theorem does not extend:

Theorem (Saksman–Seip 2009)

There exists an f in $\mathscr{H}^\infty$ such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |f(it)|^2 dt$$

does not exist. For every $\varepsilon > 0$ there is a singular inner function f in $\mathscr{H}^\infty$ such that $\|f\|_2 < \varepsilon$.

Construction of counter-examples

Again we use the Bohr correspondence and move to a polydisc. We use a beautiful construction of Rudin ("Function Theory in Polydiscs", 1969):

- If ψ is positive, bounded, and lower semi-continuous on $\mathbb{T}^m$, then ψ is a. e. the radial boundary limit of the modulus of a function in $H^\infty(\mathbb{D}^m)$.

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Choose $m = 2$, so that we only deal with two "variables", say 2^{-s} and 3^{-s}. Consider $i\mathbb{R}$ as a subset of $\mathbb{T}^2$, and cover it by an open set E of measure $< \varepsilon/2$. Take as ψ a function being 1 on E and ε outside E. The challenge is to prove that Rudin's construction leads to a function with radial limit 1 almost everywhere on $i\mathbb{R}$ (a subset of $\mathbb{T}^2$ of measure 0).

What about $\mathscr{H}^p$ when $p \neq 2, \infty$?

By the ergodic theorem, we may either define $\mathscr{H}^p$ via L^p-norms on polycircles or via L^p integral means on the imaginary axis (Bayart 2002):

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- For F a polynomial and with m_∞ Haar measure on $\mathbb{T}^\infty$,

$$\|F\|_{H^p(\mathbb{D}^\infty)} := \left(\int_{\mathbb{T}^\infty} |F(\tau)|^p \, dm_\infty(\tau) \right)^{1/p},$$

  $H^p(\mathbb{D}^\infty)$ the closure of polynomials w.r.t this norm. Use the Bohr correspondence $f \leftrightarrow F$ and set $\|f\|_{\mathscr{H}^p} := \|F\|_{H^p(\mathbb{D}^\infty)}$.

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- For f a Dirichlet polynomial,

$$\|f\|_{\mathscr{H}^p}^p = \lim_{T\to\infty} \frac{1}{T} \int_0^T |f(it)|^p \, dt;$$

  take again the closure of polynomials.

The embedding problem for $\mathscr{H}^p$

By estimates of Cole and Gamelin (1986), f in $\mathscr{H}^p$ satisfies

$$|f(\sigma + it)| \leq C(\sigma - 1/2)^{1/p}$$

for $\sigma > 1/2$, just as functions in $H^p(\mathbb{C}^+_{1/2})$.

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Question

Fix an exponent $p > 0$, not an even integer. Does there exist a constant $C_p < \infty$ such that

$$\int_0^1 \left| f\left(\frac{1}{2} + it\right)\right|^p dt \leq C_p \|f\|^p_{\mathscr{H}^p}$$

for every Dirichlet polynomial f?

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Trivial for $p = 2k$, k a positive integer, because we may apply the case $p = 2$ to f^k. The problem is probably very difficult.

Embedding problem for $p = 1$ and weak factorization

For $p = 1$, the embedding would be implied by an extension to $\mathbb{T}^\infty$ of the Ferguson–Lacey weak factorization theorem. (The problem of finding such an extension of the weak factorization theorem was raised by H. Helson (2005).)

The embedding problem for $\mathscr{H}^p$—weak version

Question

Assume that $2 < q < p < 4$. *Is it true that*

$$\left(\int_0^1 \left|f\left(\frac{1}{2}+it\right)\right|^q \, dt\right)^{1/q} \leq C_q \|f\|_{\mathscr{H}^p}$$

for every Dirichlet polynomial f*? Is this true at least for one such pair of exponents?*

Fatou theorems

Since $i\mathbb{R}$ has measure 0 when viewed as a subset of $\mathbb{T}^\infty$, care has to be taken if we want to speak about the restriction to $i\mathbb{R}$ of a function in $L^p(\mathbb{T}^\infty)$. Set for $\tau = (\tau_1, \tau_2, \dots) \in \mathbb{T}^\infty$ and $\theta \geq 0$

$$b_\theta(\tau) := (p_1^{-\theta}\tau_1, p_2^{-\theta}\tau_1, \dots).$$

The Kronecker flow on $\overline{\mathbb{D}}^\infty$:

$$T_t((z_1, z_2, \dots)) := (p_1^{-it}z_1, p_2^{-it}z_2, \dots).$$

We equip $T(z) := \{T_t(z) : t \in \mathbb{R}\}$ with the natural linear measure.

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We equip $T(z) := \{T_t(z) : t \in \mathbb{R}\}$ with the natural linear measure.

Theorem (Saksman–Seip 2009)

For every F in $H^\infty(\mathbb{D}^\infty)$ we may pick a representative $\widetilde{F}$ for the boundary function of F on $\mathbb{T}^\infty$ such that $\widetilde{F}(\tau) = \lim_{\theta\to 0^+} F(b_\theta(\tau))$ for a.e. $\tau \in \mathbb{T}^\infty$. In fact, for every $\tau \in \mathbb{T}^\infty$, we have $\widetilde{F}(\tau') = \lim_{\theta\to 0^+} F(b_\theta(\tau'))$ for a.e. $\tau' \in T(\tau)$.

Fatou theorem for $\mathscr{H}^p$

Set $\mathbb{T}^\infty_{1/2} := b_{1/2}(\mathbb{T}^\infty)$. We need to make sense of the restriction $F \mapsto F|_{\mathbb{T}^\infty_{1/2}}$ as a map from $H^p(\mathbb{D}^\infty)$ to $L^p(\mathbb{T}^\infty_{1/2})$. For F a polynomial, we must have

$$F|_{\mathbb{T}^\infty_{1/2}}(\tau) = F(b_{1/2}(\tau)).$$

Write this as a Poisson integral and use that polynomials are dense in $H^p(\mathbb{D}^\infty)$.

Theorem (Saksman–Seip 2009)

For every F in $H^p(\mathbb{D}^\infty)$ ($p \geq 2$) we may pick a representative $\widetilde{F}_{1/2}$ for the restriction $F|_{\mathbb{T}^\infty_{1/2}}$ on $\mathbb{T}^\infty$ such that $\widetilde{F}_{1/2}(\tau) = \lim_{\theta \to 1/2^+} F(b_\theta(\tau))$ for a.e. $\tau \in \mathbb{T}^\infty$. In fact, for every $\tau \in \mathbb{T}^\infty$, we have $\widetilde{F}_{1/2}(\tau') = \lim_{\theta \to 1/2^+} F(b_\theta(\tau'))$ for a.e. $\tau' \in T(\tau)$.

“Strong” ergodic theorem only when $p = 2, 4, 6, ...$?

Arguing as for $p = 2$, we now get: If F is in $H^p(\mathbb{D}^\infty)$ ($p \geq 2$) *and the embedding holds*, then for every τ in $\mathbb{T}^\infty_{1/2}$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\widetilde{F}(T_t\tau)|^p \, dt = \|\widetilde{F}_{1/2}\|^p_{L^p(\mathbb{T}^\infty)}.$$

Conversely, by the closed graph theorem (fix $T = 1$), the embedding would follow from such a “strong” ergodic theorem.

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Conversely, by the closed graph theorem (fix $T = 1$), the embedding would follow from such a "strong" ergodic theorem.

So our "strong" variant of the Birkhoff–Khinchin ergodic theorem for functions in $H^p(\mathbb{D}^\infty)$ is known to hold only when $p = 2, 4, 6, ...$!

A problem concerning Riesz projection on $\mathbb{T}^\infty$

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Riesz projection P_+ (orthogonal projection from $L^2(\mathbb{T}^\infty)$ to $H^2(\mathbb{T}^\infty)$) is obviously unbounded on L^p for any $p \neq 2$, and therefore the embedding problem can not be solved in the standard way by interpolation.

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I only know, by construction of an example, that any such p must be a little less than 4 ...