Homogeneous Operators, Jet Construction and Similarity

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Notations

- Let $\varphi_{t,a}$ for $(t, a) \in \mathbb{T} \times \mathbb{D}$ be defined by $\varphi_{t,a}(z) = t\frac{z-a}{1-\bar{a}z}, \quad z \in \mathbb{D}$.

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- Let Möb$= \{\varphi_{t,a} : (t, a) \in \mathbb{T} \times \mathbb{D}\}$ denote the bi-holomorphic automorphism group of the open unit disc $\mathbb{D}$.

Homogeneous Operator

Definition

A bounded linear operator T on a complex separable Hilbert space $\mathcal{H}$ whose spectrum is contained in $\bar{\mathbb{D}}$ is said to be *homogeneous* if $\varphi(T) = U_\varphi^* T U_\varphi$ for some unitary operator U_φ and all $\varphi \in$Möb.

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- The reproducing kernel for the tensor product $\mathbb{A}^{(\alpha)}(\mathbb{D}) \otimes \mathbb{A}^{(\beta)}(\mathbb{D})$ is

$$B^{(\alpha,\beta)}(\mathbf{z}, \mathbf{w}) := (1 - z_1\bar{w}_1)^{-\alpha}(1 - z_2\bar{w}_2)^{-\beta},$$

for $\mathbf{z} = (z_1, z_2) \in \mathbb{D}^2$ and $\mathbf{w} = (w_1, w_2) \in \mathbb{D}^2$, $\alpha, \beta > 0$.

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  for $\mathbf{z} = (z_1, z_2) \in \mathbb{D}^2$ and $\mathbf{w} = (w_1, w_2) \in \mathbb{D}^2$, $\alpha, \beta > 0$.
- We set: $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \simeq \mathbb{A}^{(\alpha)}(\mathbb{D}) \otimes \mathbb{A}^{(\beta)}(\mathbb{D})$.

A Key Proposition

Proposition (Dougalas, Misra, Varughese)

The compression of the operator $M^{(\alpha)} \otimes I$ to the ortho-complement $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_n^{(\alpha,\beta)}(\mathbb{D}^2)$ is homogeneous.

Here $\mathbb{A}_n^{(\alpha,\beta)}(\mathbb{D}^2)$ is defined by

$$\{f \in A^{(\alpha,\beta)}(\mathbb{D}^2) : {\partial_2}^{\ell} f_{|\triangle} = 0 \text{ for } 0 \le \ell \le n\}$$

where $\triangle := \{(z,z) \in \mathbb{D}^2 : z \in \mathbb{D}\}$.

A concrete realization of $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_n^{(\alpha,\beta)}(\mathbb{D}^2)$

- $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ is defined by

  $\{Jf := \sum_{i=0}^n \partial_2^i f \otimes e_i : f \in \mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)\}$, where

  $e_i, 0 \leq i \leq n$, denotes the standard unit vectors in $\mathbb{C}^{n+1}$.
- The vector space $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ inherits a Hilbert space structure via the map J.

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- $J_0^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) := \{Jf : Jf_{|_\triangle} = 0\}$.

  $\mathbb{A}_n^{(\alpha,\beta)}(\mathbb{D}^2)$ is realized in the Hilbert space $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ as $J_0^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$.

A Key Theorem

- $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}$ is defined as the set

  $\{f : f = g_{|\mathrm{res}\triangle}$ for some $g \in J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)\}$

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Theorem (Douglas, Misra, Varughese)

The compression of $M^{(\alpha)} \otimes I$ to the ortho-complement of the subspace $J_0^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ is the multiplication operator on the space $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}$.

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- $B_n^{(\alpha,\beta)}$: the reproducing kernel for the Hilbert space $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}$.
- $\mathcal{W}_n := \{M_n^{(\alpha,\beta)^*} : \alpha, \beta > 0\}, n \geq 0$: Generalized Wilkins' operators.

Two Results on $\mathcal{W}_n$

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Theorem (Misra, Shyam Roy)

The set $\mathcal{W}_n$ is a collection of irreducible homogeneous operators in $\mathrm{B}_{n+1}(\mathbb{D})$.

Some Generality

- $\mathcal{K}$: a Hilbert space with an orthonormal basis $\{e_i\}$.

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- $\mathcal{H}_{\boldsymbol{\eta}} := \oplus_{j=0}^{m} \eta_j \mathcal{H}_j$.

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- $\sum_{j=0}^{m} K_j$ is the reproducing kernel of $\mathcal{H}$
- $\sum_{j=0}^{m} |\eta_j|^2 K_j$ is the reproducing kernel of $\mathcal{H}_{\boldsymbol{\eta}}$.

The Key Proposition

Proposition

The multiplication operators on the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_{\boldsymbol{\eta}}$ are similar via the map $\iota : \mathcal{H}_{\boldsymbol{\eta}} \to \mathcal{H}$.

Another Jet Construction

- $\mathrm{Hol}(\mathbb{D}, \mathbb{C}^k)$: the space of all holomorphic functions taking values in $\mathbb{C}^k$, $k \in \mathbb{N}$
- $2\lambda_j := 2\lambda - m + 2j$, where $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}$ with $2\lambda - m > 0$, $0 \le j \le m$.

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- $2\lambda_j := 2\lambda - m + 2j$, where $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}$ with $2\lambda - m > 0$, $0 \leq j \leq m$.
- The operator $\Gamma_j : \mathbb{A}^{(2\lambda_j)}(\mathbb{D}) \longrightarrow \mathrm{Hol}(\mathbb{D}, \mathbb{C}^{m+1})$ by the formula

$$(\Gamma_j f)(\ell) = \begin{cases} \binom{\ell}{j} \frac{1}{(2\lambda_j)_{\ell-j}} f^{(\ell-j)} & \text{if } \ell \geq j \\ 0 & \text{if } 0 \leq \ell < j, \end{cases}$$

for $f \in \mathbb{A}^{(2\lambda_j)}(\mathbb{D})$, $0 \leq j \leq m$, where

$(x)_n := x(x+1)\cdots(x+n-1)$.
$(\Gamma_j f)(\ell) :=$ the ℓ-th component of the function $\Gamma_j f$
$f^{(\ell-j)}$:=the $(\ell-j)$-th derivative of f.

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- $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D}) := \oplus_{j=0}^{m} \mu_j \mathbf{A}^{(2\lambda_j)}(\mathbb{D}),\ 1 = \mu_0, \mu_1, \ldots, \mu_m > 0.$

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- $\mathbf{B}^{(\lambda,\boldsymbol{\mu})} = \sum_{j=0}^{m} \mu_j^2 \mathbf{B}^{(2\lambda_j)}$: the reproducing kernel for $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$.
- $M^{(\lambda,\boldsymbol{\mu})}$: the multiplication operator on the Hilbert space $\mathbf{A}^{(\lambda,\mu)}(\mathbb{D})$.

A Large Class of Irreducible Homogeneous Operators

- $H_{1,m+1}(\mathbb{D}) := \{M^{(\lambda,\boldsymbol{\mu})^*}, \lambda > \frac{m}{2}, \boldsymbol{\mu} > 0\} \subseteq \mathrm{B}_{m+1}(\mathbb{D})$

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Irreducibility of members of $H_{1,n+1}(\mathbb{D})$ can be proved easily by using irreducibility of members of $\mathcal{W}_n$.

Theorem (Alternative proof)

The multiplication operator $M^{(\lambda,\boldsymbol{\mu})}$ on the Hilbert space $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$ is irreducible.

Steps of the proof

- $\mathcal{H}_{\boldsymbol{\eta}} := \mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$, with $\eta_j := \frac{\boldsymbol{\mu}(j)}{\boldsymbol{\mu}_0(j)}$ for $0 \le j \le m$.

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- $\mathcal{H} := \mathbf{A}^{(\lambda,\boldsymbol{\mu}_0)}(\mathbb{D})$
- $\iota : \mathcal{H}_{\boldsymbol{\eta}} \longrightarrow \mathcal{H}$ satisfies $\iota M^{(\lambda,\boldsymbol{\mu})} = M^{(\lambda,\boldsymbol{\mu}_0)}\iota$.

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- $\mathcal{H} := \mathbf{A}^{(\lambda,\boldsymbol{\mu}_0)}(\mathbb{D})$
- $\iota : \mathcal{H}_{\boldsymbol{\eta}} \longrightarrow \mathcal{H}$ satisfies $\iota M^{(\lambda,\boldsymbol{\mu})} = M^{(\lambda,\boldsymbol{\mu}_0)}\iota$.
- $\iota^*, \iota^{-1} : \mathcal{H} \longrightarrow \mathcal{H}_{\boldsymbol{\eta}}$ are given by $\iota^*(f) = \iota^{-1}(f) = f$ for $f \in \mathcal{H}$.

Steps of the proof

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THANK YOU.