Extremals for families of commuting operators.

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Topic: Dilations and extensions of d-tuples of commuting Hilbert space operators

Example for $d = 1$:

Thm 1. *(the Sz.-Nagy dilation theorem)*
$T \in \mathcal{B}(\mathcal{H}), \|T\| \leq 1$
$\Rightarrow \exists\ V \in \mathcal{B}(\mathcal{K}), \quad \mathcal{H} \subseteq \mathcal{K},$
$\qquad V\mathcal{H} \subseteq \mathcal{H}, \quad \|V^*x\| = \|x\|,$
$\qquad T = V|\mathcal{H}.$

$V =$ co-isometric extension of T

$V = S^* \oplus U,$

S unilateral shift of some multiplicity,
U unitary

All Hilbert spaces in the following are supposed to be separable.

$d \in \mathbb{N}, \quad \mathbb{B}^d = \{z \in \mathbb{C}^d : |z| < 1\}$

Defn 2. *(Agler) A* **family** *$\mathcal{F}$ is a collection of d-tuples $T = (T_1, .., T_d)$ of Hilbert space operators, $T_i \in \mathcal{B}(\mathcal{H})$ such that*

(a) $\mathcal{F}$ is bounded,
$\exists c > 0 \ \forall T = (T_1, .., T_d) \in \mathcal{F} : \|T_i\| \leq c \quad \forall i$

(b) restrictions to invariant subspaces
$T \in \mathcal{F}, \mathcal{M} \subseteq \mathcal{H}, T_i\mathcal{M} \subseteq \mathcal{M} \ \forall i \Rightarrow T|\mathcal{M} \in \mathcal{F}$

(c) direct sums
$T_n \in \mathcal{F} \Rightarrow \oplus_n T_n \in \mathcal{F}, \quad T_n = (T_{1n}, .., T_{dn})$

(d) unital $*$ -representations
$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}), \pi(I) = I,$
$T = (T_1, .., T_d) \in \mathcal{F}$
$\Rightarrow \pi(T) = (\pi(T_1), .., \pi(T_d)) \in \mathcal{F}.$

Examples:

$d = 1$:

$\mathcal{F} =$ contractions, $T^*T \leq I$

isometries, $T^*T = I$

subnormal contractions

hyponormal contractions

$d \geq 1$:

$\mathcal{F} =$ contractions

commuting contractions

isometries

commuting isometries

$\mathcal{F} =$ commuting spherical contractions

commuting row contractions (d-contractions)

commuting spherical isometries

Defn 3. *If* $T = (T_1, .., T_d), T_i \in \mathcal{B}(\mathcal{H})$,
$S = (S_1, .., S_d), S_i \in \mathcal{B}(\mathcal{K})$, $T, S \in \mathcal{F}$,
then

$$T \leq S \Leftrightarrow \mathcal{H} \subseteq \mathcal{K}, S\mathcal{H} \subseteq \mathcal{H}, T = S|\mathcal{H}$$
$$\Leftrightarrow S = \begin{pmatrix} T & X \\ 0 & Y \end{pmatrix}$$
$$\Leftrightarrow S \text{ extends } T.$$

Defn 4. T *is* **extremal** *for* $\mathcal{F}$,
$$\Leftrightarrow S \geq T, S \in \mathcal{F} \Rightarrow S = \begin{pmatrix} T & 0 \\ 0 & Y \end{pmatrix} = T \oplus Y.$$

We will write $T \in \text{ext}(\mathcal{F})$

Thm 5. *(Agler)*
$\mathcal{F}$ *family,* $T \in \mathcal{F} \Rightarrow \exists\ S \in \text{ext}(\mathcal{F})\ S \geq T$

Examples:

$\mathcal{F}_c =$ contractions $\Rightarrow$ ext$(F_c) =$ co-isometries
If

$$\|T\| \leq 1, x \neq 0, \quad S = \begin{pmatrix} T & \varepsilon x \\ 0 & b \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathbb{C})$$

then $\|S\| \leq 1$ for suff. small $\varepsilon > 0$

$$\Leftrightarrow$$

$$|b| < 1 \text{ and } x \in \operatorname{ran} D_*, \quad D_* = (I - TT^*)^{1/2}$$

Thus, nonextremal contractions have non-trivial rank 1 extensions.

Cor 6. *(Sz. Nagy)*
Every contraction has a co-isometric extension.

$\mathcal{F}_i$ = isometries $\Rightarrow$ ext(F_i) = unitaries

$T \in \mathcal{F}_i \Rightarrow T = S \oplus U$

Thus, nonextremal isometries have nontrivial rank 1 extensions.

Cor 7. *Every isometry is subnormal.*

Commuting spherical isometries (A. Athavale)

$$\mathcal{F}_{si} = \{T = (T_1, .., T_d) : T_i \leftrightarrow T_j,$$
$$\textstyle\sum_{i=1}^{d} \|T_i x\|^2 = \|x\|^2 \quad \forall x \in \mathcal{H}\}$$

$$\text{ext}(\mathcal{F}_{si}) = \{U = (U_1, .., U_d) : U_i \leftrightarrow U_j, U_i \text{ normal}$$
$$\textstyle\sum_{i=1}^{d} \|U_i x\|^2 = \|x\|^2 \quad \forall x \in \mathcal{H}\}$$

$= \text{commuting spherical unitaries}$

One direction in the proof follows Attele-Lubin, JFA, 1996.

Cor 8. *(Athavale, 91) Every commuting spherical isometry is jointly subnormal.*

Thm 9. *Let $T \in \mathcal{F}_{si}$. Then*

T has a nontrivial rank one extension in $\mathcal{F}_{si}$ if and only if

there exists $b \in \mathcal{B}_d$ such that the Koszul complex for $T - b$ is not exact at $\Lambda^1(\mathcal{H})$.

OR equivalently

T has only trivial rank one extensions in $\mathcal{F}_{si}$ if and only if

whenever $b = (b_1, .., b_d) \in \mathcal{B}_d$, $x_1, .., x_d \in \mathcal{H}$ with $(T_i - b_i)x_j = (T_j - b_j)x_i$, then $\exists\ x \in \mathcal{H}$ such that $x_i = (T_i - b_i)x$.

Examples: If $d > 1$, then $(M_z, H^2(\partial\mathcal{B}_d))$ has no nontrivial rank one extensions in $\mathcal{F}_{si}$.

If $S = (M_z, H^2(\partial\mathbb{D}))$, then $(S, 0, .., 0) \in \mathcal{F}_{si}$ has nontrivial rank one extensions in $\mathcal{F}_{si}$.

Commuting spherical contractions

Drury 78, Mueller-Vasilescu 93,
Arveson 98

$$\mathcal{F}_{sc} = \{T = (T_1, .., T_d) : T_i \leftrightarrow T_j, \\ \textstyle\sum_{i=1}^d \|T_i x\|^2 \leq \|x\|^2 \quad \forall x \in \mathcal{H}\}$$

$T \in \mathcal{F}_{sc} \;\Leftrightarrow\; \sum_{i=1}^d T_i^* T_i \leq I$, commuting

$\text{ext}(\mathcal{F}_{sc}) = \{S^* \oplus U\}$

$U =$ commuting spherical unitary
$S =$ d-shift of some multiplicity

$S = M_z$ on $H_d^2(\mathcal{D}) = H_d^2 \otimes \mathcal{D}$

$H_d^2 \subseteq$ Hol$(\mathbb{B}^d)$, Drury-Arveson-Hardy space defined by reproducing kernel

$$k_w(z) = \frac{1}{1 - \langle z, w\rangle}, \quad z, w \in \mathbb{B}^d$$

Thm 10. *(Richter-Sundberg) Let* $T = (T_1, ..T_d)$ *be a commuting operator tuple.*

Then the following are equivalent

(a) $T \in$ ext$(\mathcal{F}_{sc})$

(b) $T = S^* \oplus U$

(c) (1) $\sum_{i=1}^{d} T_i^* T_i = P =$ *a projection*
(2) If $x_1, .., x_d \in \mathcal{H}$ *with* $T_i x_j = T_j x_i$*,*
then $\exists x \in \mathcal{H}$ *with* $x_i = T_i x$*.*
(3) $\sum_{i=1}^{d} T_i T_i^* \geq I$*.*

(c2) says that the Koszul complex for T is exact at $\Lambda^1(\mathcal{H})$.

Note 1: For $d = 1$ (c) becomes

(1) $T^*T = P$, i.e. T is a partial isometry

(2) if $x_1 \in \mathcal{H}$, then $\exists x \in \mathcal{H}$ with $x_1 = Tx$, i.e. T is onto

(3) $TT^* \geq I$, so T is onto

Hence (1)&(2) or (1)&(3) are equivalent to T^* being an isometry.

For $d > 1$ let $T = M_z$ on $H^2(\partial\mathbb{B}^d)$, then (c1) and (c2) are satisfied, but (c3) is not.

If $\mathcal{M} = \{f \in H^2_d : f(0) = 0\}$, $P : H^2_d \to \mathcal{M}$ projection, then

$$T = PS^*|\mathcal{M}$$

satisfies (c1) and (c3), but not (c2).
(c1) $\sum_{i=1}^d T_i^* T_i = P - \sum_i z_i \otimes z_i$

Prop 12. *Let* $T = (T_1, ..T_d)$ *be a commuting operator tuple such that*

(1) $\sum_{i=1}^{d} T_i^* T_i = P =$ *a projection*

(2) If $x_1, .., x_d \in \mathcal{H}$ *with* $T_i x_j = T_j x_i$*, then* $\exists x \in \mathcal{H}$ *with* $x_i = T_i x$.

Then

$$T = S^* \oplus V$$

$S =$ *d-shift of some multiplicity*
$V =$ *spherical isometry*

Commuting row contractions (d-contractions)

$\mathcal{F}_{rc} = \{T : T^* \in \mathcal{F}_{sc}\}$

$= \{T : T_i \leftrightarrow T_j, \sum_{i=1}^d \|T_i^* x\|^2 \leq \|x\|^2 \forall x\}$

$= \{T : T_i \leftrightarrow T_j, \|\sum_{i=1}^d T_i x_i\|^2 \leq \sum_{i=1}^d \|x_i\|^2 \ \ \forall x_i\}$

$T \in \mathcal{F}_{rc}$
$\Leftrightarrow (T_1, .., T_d) : \mathcal{H} \oplus .. \oplus \mathcal{H} \to \mathcal{H}$ is contractive
commutative

$\Rightarrow T^* = S^* \oplus U|\mathcal{H}$
(by Mueller/Vasilescu-Arveson)

$\Rightarrow T = P_{\mathcal{H}}(S \oplus U^*)|\mathcal{H}$,
$\mathcal{H}$ =co-invariant for $S \oplus U^*$

ext(F_{rc}) =?

Lemma 13. *Let $T \in \mathcal{F}_{rc} \cap \mathcal{B}(\mathcal{H})^d$, and let $D_* = (I - \sum_{i=1}^{d} T_i T_i^*)^{1/2}$.*

Then T has a nontrivial rank one extension in $\mathcal{F}_{rc}$, if and only if
T has a nontrivial finite rank extension in $\mathcal{F}_{rc}$, if and only if
$\exists \quad b = (b_1, ..., b_d) \in \mathcal{B}_d$, $x_1, ..., x_d \in \mathcal{H}$ *such that*

(1) $\sum_{i=1}^{d} \|x_i\|^2 \neq 0$,

(2) $(T_i - b_i)x_j = (T_j - b_j)x_i$ *for all i, j, and*

(3) $x_i \in \mathrm{ran} D_*$ *for each i.*

$$S_i = \begin{pmatrix} T_i & \varepsilon x_i \\ 0 & b_i \end{pmatrix}, \qquad \varepsilon > 0 \text{ small}$$

<u>Rmk:</u> $\exists$ non-extremals with only trivial finite rank extensions in $\mathcal{F}_{rc}$.

$$D_* = (I - \sum_{i=1}^{d} T_i T_i^*)^{1/2}$$

Thm 14. *(easy)*
(a) If $D_* = 0$, then $T \in$ ext(F_{rc})
spherical co-isometries

(b) If D_* is onto, then $T \notin$ ext(F_{rc}).

(c) If D_* is a projection, then

$T \notin$ *ext(F_{rc})*
$\Leftrightarrow \exists x_1, .., x_d \in$ *ran* $D_*, \sum_{i=1}^{d} \|x_i\|^2 > 0$
with $T_i x_j = T_j x_i$

(c) $\Rightarrow$ $S \in$ ext($\mathcal{F}_{rc}$) - the d-shift
$D_* =$ projection onto constants

$$D_* = (I - \sum_{i=1}^{d} T_i T_i^*)^{1/2}$$

Thm 15. *(R-S)*
If $T \in \mathcal{F}_{rc}$ and if D_* has rank one, i.e.

$$D_* = u \otimes u$$

for some $u \neq 0$, then

$T \in$ *ext*$(F_{rc}) \Leftrightarrow$ *dim span*$\{u, T_1 u, .., T_d u\} \geq 3$

$$S_i = \begin{pmatrix} T_i & A_i \\ 0 & B_i \end{pmatrix} \in \mathcal{F}_{rc} \Rightarrow$$

ran$A_i \subseteq$ ranD_* and
$T_i A_j - T_j A_i = A_j B_i - A_i B_j$

If $S = (M_z, H_d^2) =$ the d-shift,
if $\mathcal{M}$ is invariant for S, $\mathcal{M} \neq H_d^2$
then

$$T = P_{\mathcal{M}^\perp} S | \mathcal{M}^\perp \in \mathcal{F}_{rc},$$

and D_* has rank 1:

$$\begin{aligned} D_*^2 &= I_{\mathcal{M}^\perp} - \sum_{i=1}^{d} P_{\mathcal{M}^\perp} S_i S_i^* P_{\mathcal{M}^\perp} \\ &= P_{\mathcal{M}^\perp}(I - \sum S_i S_i^*) P_{\mathcal{M}^\perp} \\ &= u \otimes u, \end{aligned}$$

$$u = P_{\mathcal{M}^\perp} 1 \neq 0$$

Cor 16. *If* $\mathcal{M} \neq H_d^2$, *if*

$$\mathcal{L} = \{a + \sum_{i=1}^{d} b_i z_i\},$$

then $T = P_{\mathcal{M}^\perp} S|\mathcal{M}^\perp \notin$ *ext*(F_{rc}) *if and only if*

$$\textit{dim } \mathcal{M} \cap \mathcal{L} \in \{d-1, d\}$$

In fact, in this case, if $T \notin$ ext(F_{rc}), then

$$T_i = a_i I + b_i S$$

for some S with rank $(I - SS^*) = 1$.

The Theorem can be used to produce finite dimensional examples

$T \in$ ext(F_{rc}) but $D_* \neq$ a projection.