SEMICROSSED PRODUCTS WITH TRANSFER OPERATORS

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1. Introduction

Let X be a compact metric space, and $\mathcal{P}$ an abelian semigroup with identity 0 such that $\mathcal{P}-\mathcal{P}$ is a group G.

We also require that the semigroup act on the space X such that for each $n \in \mathcal{P}$ there is a map $\varphi_n : X \to X$ which is continuous and surjective.

The semigroup $\mathcal{P}$ is also a semigroup of endomorphisms α_n on $C(X)$ given by $\alpha_n(f) = f \circ \varphi_n$. Furthermore, we assume there is a continuous map $\mathcal{L}_n : C(X) \to C(X)$ which act as a left inverse of α_n, $n \in \mathcal{P}$. The map $\mathcal{L}_n$ is the *transfer operator*.

We construct an operator algebra $\mathcal{A}(X, \mathcal{P})$ as follows: let $\mathcal{A}_0$ be the formal algebra generated by $C(X)$ and the symbols S_n, $n \in \mathcal{P}$ subject to the relations

$$S_n f = \alpha_n(f) S_n \quad \text{and} \quad f S_n = S_n \mathcal{L}_n(f) \tag{1}$$

for all $f \in C(X)$ and all $n \in \mathcal{P}$.

The *admissible representations* of $\mathcal{A}_0$ are representations ρ of $\mathcal{A}_0$ into the bounded operators on a Hilbert space such that

(1) ρ restricted to $C(X)$ is a *-representation;
(2) $\rho(S_n)$ is an isometry;
(3) the representation ρ respects the relations (1).

The semicrossed product of Exel type is the completion of the algebra $\mathcal{A}_0$ with respect to the operator norm

$||a|| = \sup\{||\rho(a)|| : \rho$ is an admissible representation

for $a \in \mathcal{A}_0$.

We can compare the (standard) semicrossed product, with the semicrossed product of Exel type, by looking at the groupoid associated with the C*-envelope of the semicrossed product. If we take a single map φ (and the semigroup which it generates) which admits a transfer operator, then the groupoid associated with the standard semicrossed product is a transformation groupoid. Thus, there is a compact metric space Y and a homeomorphism $\psi : Y \to Y$ and a continuous surjection $p : Y \to X$ such that

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & Y \\ p\Big\downarrow & & p\Big\downarrow \\ X & \xrightarrow{\varphi} & X \end{array} \qquad (\dagger)$$

The transformation groupoid $\mathfrak{G}$ consists of triples

$$(y, n, z) \text{ with } y, z \in Y,\ n \in \mathbb{Z}$$

such that $\psi^n(y) = z$.

By contrast, the groupoid associated with the the semicrossed product of Exel type consists of

$$(x, n, y) \text{ with } x, y \in X,\ n \in \mathbb{Z}$$

for which there exists nonnegative integers $k,\ j$ with $k - j = n$ and $\varphi^k(x) = \varphi^j(y)$. This is the *Deaconu-Renault* groupoid.

The difference between the two groupoids is a reflection of the fact that the there are two conditions on representations which must be satisfied for the Exel semicrossed products, whereas only one condition for the standard crossed products. Furthermore, the condition that the representation respect the transfer operator is especially restrictive.

The next logical question is, in what circumstances does there exist a transfer operator? Indeed, such operators do not generally exit for semigroups. Exel and Renault considered the case where $\mathcal{P}$ is a semigroup of local homeomorphism (with certain additional conditions).

I wanted to be able to work in a broader context. Thus, for each $n \in \mathcal{P}$, in addition to being continuous and surjective, I assume that φ_n is locally injective. Local injectivity, without requiring that the map be open, is a natural generalization of the local homeomorphism setting. But even for singly generated semigroups these conditions do not guarantee the existence of a transfer operator.

Definition 1. We will say ω is a *cocyle* on a dynamical system $(\mathcal{P}, X)$ if

(1) ω is a function from $\mathcal{P} \times X \to \mathbb{R}$, and $\omega(n, x) \geq 0$ for all $(n, x) \in \mathcal{P} \times X$;
(2) for each $y \in X,\ n \in \mathcal{P},\ \sum_{\varphi_n(x)=y} \omega(n, x) = 1$;
(3) for each $n \in \mathcal{P}$, the map $x \in X \to \omega(n, x)$ is continuous;

(4) ω satisfies the cocycle identity:

$$\omega(m+n,x) = \omega(m,x)\omega(n,\varphi_m(x)).$$

A dynamical system $(\mathcal{P}, X)$ will be called *admissible* if it admits a cocycle.

The existence of a cocycle provides for a transfer operator, by means of

$$\mathcal{L}_n(f)(y) = \sum_{\varphi_n(x)=y} \omega(n,x)f(x),\ \ f \in C(X).$$

It is easy to check that this gives a left inverse of the endomorphism α_n.

Define a representation $\pi : \mathcal{A}_0 \to \mathcal{B}(\ell_2(X))$ by:

$$\pi(f)\xi = f\xi \text{ and } \pi(S_n)\xi = \sqrt{\omega(n,x)}\xi \circ \varphi_n,$$

$n \in \mathcal{P}$. More generally, given a character γ of G one can define a representation π_γ of $\mathcal{A}_0$ which acts as before on functions in $C(X)$ and on elements S_n by

$$\pi_\gamma(S_n)\xi =< \gamma, n > \sqrt{\omega(n,x)}\xi \circ \varphi_n.$$

2. CSLI MAPS

Lemma 1. *Let (X,φ) be a CSLI dynamical system. Then for all $x \in X$, $|\varphi^{-1}(x)| < \infty$.*

Corollary 1. *The set $\{u \in X :\ |\varphi^{-1}(u)| \leq N\}$ is open.*

The next two results concern singly generated semigroups. CSLI referes to continuous, surjective and locally injective.

Corollary 2. *A necessary condition for a CSLI system (X, φ) to admit a cocycle is: for every $y \in X$ there exists a point $x \in \varphi^{-1}(y)$ such that φ is locally open at x.*

Corollary 3. *Let (X, φ) be a CSLI system. Then the system admits a strictly positive cocycle if and only if φ is a local homeomorphism.*

The following is a CSLI map which does not admit a cocycle:

Example 1. This is an example of a semigroup $\mathcal{P} = \mathbb{N}$ of a CSLI dynamical system which is not admissible.

Let $\varphi : \Pi_{n\in\mathbb{Z}}\mathbb{T}_n \to \Pi_{n\in\mathbb{Z}}\mathbb{T}_n$ where $\mathbb{T}_n = \mathbb{T} = [0, 1)$, as follows: for a point $\mathbf{x} = (x_n)$ in the product space, set

$$\varphi(\mathbf{x}) = \mathbf{y} \text{ where } y_{n-1} = x_n$$

for all $n \neq 1$ and $y_0 = 2x_1\ (mod 1)$. Note that φ is a local homeomorphism of $\Pi_{n\in\mathbb{Z}}\mathbb{T}_n$.

Let $Z \subset \Pi_{n\in\mathbb{Z}}\mathbb{T}_n$ consist of those sequences $\mathbf{x} = (x_n)$ satisfying: for all $n \geq 1,\ 0 \leq x_n \leq \frac{1}{2}$. Clearly Z is closed, and $\varphi(Z) \subset Z$. We take

$$X = \cap_{n=0}^{\infty}\varphi^n(Z),$$

where φ^0 is the identity, and for $n > 1,\ \varphi^n$ is the n-fold composition of φ with itself. Then $\varphi(X) = X$.

Changing notation so φ refers to the restriction of φ to X, the dynamical system (X, φ) is CSLI.

To show that (X,φ) is not admissible, we suppose to the contrary that ω is a cocycle for φ. Set

$$\mathbf{y}^0 = (\dots, 0, \underset{0}{0}, \frac{1}{2}, \frac{1}{2}, \dots)$$

where the underset 0 denotes the 0-th position in the array. Note that $\varphi^{-1}(\mathbf{y}^0) = \{\mathbf{y}^0, \mathbf{w}^0\}$ where

$$\mathbf{w}^0 = (\dots, 0, \underset{0}{0}, 0, \frac{1}{2}, \frac{1}{2}, \dots).$$

First we show that $\omega(\mathbf{y}^0) = 1$. To this end, define

$$\mathbf{y}(t) = (\dots, t, \underset{0}{t}, \frac{t}{2}, \frac{t}{2}, \dots)$$

and note that, for $t > 0$, $\varphi^{-1}(\mathbf{y}(t)) = \{\mathbf{y}(t)\}$. Hence $\omega(\mathbf{y}(t)) = 1$ for $t > 0$. Since $\mathbf{y}(t) \to \mathbf{y}^0$ as $t \to 1$ continuity of ω forces $\omega(\mathbf{y}^0) = 1$.

Next we claim that $\omega(\mathbf{w}^0) = 1$. To this end, we define

$$\mathbf{u}(t) = (\dots, t, \underset{0}{t}, \frac{1}{2} - t, \frac{1}{2} - t, \dots)$$

for $0 < t < \frac{1}{2}$. To see that $\mathbf{u}(t) \in X$, note first that $\mathbf{u}(t) \in Z$. Let $n \in \mathbb{N}$, $n \geq 1$ and set

$$\mathbf{w}(t) = (\dots, t, \underset{0}{t}, \frac{t}{2}, \dots, \underset{n}{\frac{t}{2}}, \frac{1}{2} - t, \frac{1}{2} - t, \dots).$$

Then $\mathbf{w}(t) \in Z$ and $\varphi^n(\mathbf{w}(t)) = \mathbf{u}(t)$. Thus, $\mathbf{u}(t) \in \varphi^n(Z)$ for every $n \geq 0$, so $\mathbf{u}(t) \in X$. Now set $n = 1$. The same argument shows that $\mathbf{w}(t) \in X$, and furthermore $\mathbf{w}(t)$ is the single inverse image of $\mathbf{u}(t)$.

Thus, $\omega(\mathbf{w}(t)) = 1$. As $t \to 0$, $\mathbf{w}(t) \to \mathbf{w}^0$. Continuity forces $\omega(\mathbf{w}^0) = 1$. But the cocycle condition $\omega(\mathbf{y}^0) + \omega(\mathbf{w}^0) = 1$ is violated, so no cocycle exists and the system (X, φ) is not admissible, and in particular, φ is not a local homeomorphism.

Also, there are dynamical systems which are CSLI but not local homeomorphisms, which are admissible.

Remark 1. Define the conditional expectation

$$E(f)(x) = \alpha \circ L(f)(x) = \sum_{\varphi(u)=\varphi(x)} \omega(u) f(u).$$

Then if ω is not strictly positive, the conditional expectation can be degenerate. Indeed, suppose $\omega(x) = 0$ in a neighborhood $\mathcal{U}$ of a point x_0. Suppose f is a nonnegative function supported in $\mathcal{U}$ and that φ is injective on $\mathcal{U}$. Then for $x \in X$

$$\begin{aligned} E(f)(x) &= \sum_{\varphi(t)=\varphi(x)} f(t)\omega(t) \\ &= 0 \end{aligned}$$

since ω is zero where f is nonzero.

Thus, in some cases the conditional expectation associated to the cocycle is degenerate. However, it can happen that the cocycle vanishes but that the conditional expectation is nondegenerate

Definition 2. Let $a_1, \ldots, a_k$ be elements of an abelian semigroup $\mathcal{P}$. This set will be called *independent* if for any nonempty subset $E \subset \{1, \dots, k\}$ and non-negative integers $n_1, \dots, n_k$ the relation

$$\sum_{j \in E} n_j a_j = \sum_{j \in E^c} n_j a_j$$

implies

$$n_1 = \cdots = n_k = 0.$$

In case the complement $E^c = \emptyset$, we interpret the right side of the equation to be zero.

Let $\mathcal{P}$ be an abelian semigroup isomorphic with $\mathbb{N}^k$, and let $e_1, \dots, e_k$ be a set of independent generators of $\mathcal{P}$.

Proposition 1. *Let $\mathcal{P}$ act on the compact metric space X. Then the the action is admissible iff each φ_{e_j} is an admissible action, $1 \leq j \leq k$.*

3. Divisible Semigroups

An abelian group $\mathcal{G}$ is *divisible* if the equation $mx = a$ $(m \in \mathbb{N},\ a \in \mathcal{G})$ has a solution $x \in \mathcal{G}$. One could use the same definition for semigroups. However, we want to consider examples such as the semigroup $\mathcal{P}$ of positive dyadic rationals. Let $\mathcal{D} = \{\frac{k}{2^n},\ k \in \mathbb{Z},\ n \in \mathbb{Z}\}$ be the group of dyadic rationals. This is not divisible, as the equation $mx = a$ is solvable for $x \in \mathcal{D}$ only for m a power of 2. Thus, for our purposes an alternative definition is appropriate.

Definition 3. A sequence $\{d_k\}$ in a semigroup $\mathcal{P}$ will be called a *fundamental sequence* if

(1) there exists a sequence of integers $n_k > 1$ such that $d_k = n_k d_{k+1},\ k \geq 1$, and
(2) For every $d \in \mathcal{P}$ there exists $k \in \mathbb{N}$ such that d_k divides d.

We say $\mathcal{P}$ is *divisible* if it contains a fundamental sequence.

Proposition 2. *Let $\mathcal{P}$ be a divisible semigroup of CSLI maps on X. Then either all $\varphi_d,\ d \in \mathcal{P}$, are homeomorphisms, or else none is a homeomorphism.*

Theorem 1. *Let $\mathcal{P}$ be a divisible semigroup of CSLI maps acting on a compact metric space X. Suppose $\mathcal{P}$ separates the points of X. Then $\mathcal{P}$ consists of homeomorphisms.*

It is not *a priori* obvious that divisible semigroups of CSLI maps which are not homeomorphisms exist. Before constructing the example, we remind the reader of a construction which has been used to "cut up" the real numbers to obtain a zero-dimensional space, Z. For each dyadic rational $d \in \mathbb{R}$, we replace d by two points d^- and d^+ so that $d^- < d^+$ and no point lies between $d^-,\ d^+$. Thus Z is an ordered set. Now we introduce a topology by taking as a base $\mathcal{B}$ for the topology the sets $[r^+, s^-]$ where $r < s$ are dyadic rationals. In this topology, every "open interval" $(a, b) = \{x \in Z,\ a < x < b\}$ is an open set in the topology of Z.

Observe that the complement of an interval $[r^+, s^-]$ is also open, so that $[r^+, s^-]$ is closed, hence clopen. One can show that the closed intervals $[a, b]$, $a < b \in Z$ are compact. Thus Z is a locally compact Hausdorff space, which is metrizable, as the base $\mathcal{B}$ is countable.

Example 2. This is an example of a divisible semigroup. We construct a compact metric space $X = X_1 \cup X_2 \cup X_3$, the union of three disjoint sets. Take

$$X_1 = [0^+, +\infty]$$

the one-point compactification of the interval $z \in Z : \ z \geq 0^+$. Now we take X_2, X_3 both to be the one-point compactifications of copies of $(-\infty, 0^-]$ in Z. To distinguish them, we use superscripts hat and tilde. Thus,

$$X_2 = [-\hat{\infty}, \hat{0}^-] \text{ and } X_3 = [-\tilde{\infty}, \tilde{0}^-].$$

The set Z is not a group under addition, but there is an action of the group $\mathcal{D}$ of dyadic rationals on Z, as follows: let $d \in \mathcal{D}$ and define translation φ_d on Z by

$$\varphi_d(x) = \begin{cases} d + x \text{ if } x \text{ is not a dyadic rational;} \\ (d + x)^+ \text{ if } x = r^+ \text{ where } r \text{ is a dyadic rational} \\ (d + x)^- \text{ if } x = r^- \text{ where } r \text{ is a dyadic rational.} \end{cases}$$

It is easy to see that φ_d is continuous, as φ_d^{-1} maps basic open intervals to basic open intervals. Similarly, φ_d is seen to be an open map. Since it is both injective and surjective, it is a homeomorphism of Z.

Let $\mathcal{P}$ denote the positive dyadic rationals, and, changing notation, let φ_d $(d \in \mathcal{P})$ denote an action of X, which we define as follows: φ_d leaves all three points at infinity fixed. For $x \in X_1 \cup X_2$ not a point at infinity, we let $\varphi_d(x)$ be defined as follows: if $x \in X_2,\ x = \hat{z}$ for some $z \in Z$, and $d \in \mathcal{P}$,

$$\varphi_d(x) = \begin{cases} \widehat{d+z} \in X_2 \text{ if } d+z \leq 0^- \\ d+z \in X_1 \text{ if } d+z \geq 0^+. \end{cases}$$

If $x \in X_1$, then $\varphi_d(x)$ is defined exactly as on Z. φ_d acts similarly on $X_1 \cup X_3$. Clearly, φ_d is surjective on X. And in the same way as with Z, one sees that φ_d is continuous and open. Note that if $x \in [-d^+, 0^-] \subset Z$ that $\varphi_d(\hat{x}) = \varphi_d(\tilde{x})$, so that φ_d is not one-to-one. Thus, $\{\varphi_d : \ d \in \mathcal{P}\}$ is a semigroup of local homeomorphisms on the compact space X which are not homeomorphisms. Note that the semigroup $\mathcal{P}$ has a fundamental sequence, namely $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$, so that $\mathcal{P}$ is a divisible semigroup.

Next we show that the semigroup is admissible. To simplify notation, when $x \in X$ belongs to either X_2 or X_3, and there is no need to distinguish between $X_2,\ X_3$, we will omit the superscripts hat and tilde.

Define the cocycle ω on $\mathcal{P} \times X$ by

$$\omega(d,x) = \begin{cases} 1, \text{ if } x \le -d^- \text{ or } x \ge 0^+ \\ \frac{1}{2} \text{ if } -d^+ \le x \le 0^-. \end{cases} \tag{2}$$

We do this for all $d \in \mathcal{P}$ and $x \in X$. We need to show this is consistent with the cocycle identity. So suppose $e,\ f \in \mathcal{P}$ with $e+f = d$. Suppose $x \le -d^-$. Then $\omega(e,x) = 1$. Now we claim that $\varphi_e(x) \le -f^-$. For otherwise, we would have $\varphi_e(x) \ge -f^+$, hence $\varphi_f(\varphi_e(x)) \ge \varphi_f(-f^+) \ge 0^+$. But then $\varphi_d(x) = \varphi_f(\varphi_e(x)) \ge 0^+$ which contradicts that $x \le -d^-$. Thus, both $\omega(e,x)$ and $\omega(f,\varphi_e(x))$ equal 1, as does $\omega(d,x)$.

The case were $x \ge 0^+$ is easier, for then it is clear that $\omega(e,x)$ and $\omega(f,\varphi_e(x))$, and $\omega(d,x)$ are all equal to 1.

Finally, let $-d^+ \le x \le 0^-$. Consider two cases: if $\varphi_e(x) \ge 0^+$, then $|\varphi_e^{-1}(\varphi_e(x))| = 2$, so that $\omega(e,x) = \frac{1}{2}$. As $\varphi_e(x) \ge 0^+$, $\omega(f,\varphi_e(x)) = 1$. By definition $\omega(d,x) = \frac{1}{2}$, so that the equality

$$\omega(x,d) = \omega(e,x)\omega(f,\varphi_e(x))$$

holds. In the other case, $\varphi_e(x) \le 0^-$, that is,

$$-d^+ \le x \le \varphi_e(x) \le 0^-.$$

Then $\omega(e,x) = 1$. But $\varphi_f(\varphi_e(x)) = \varphi_d(x) \ge 0^+$ so that both $\omega(f,\varphi_e(x))$, and $\omega(d,x) = \frac{1}{2}$. So again the cocyle identity

$$\omega(d,x) = \omega(e,x)\omega(f,\varphi_e(x))$$

holds.

Finally we need to observe that for arbitrary $d \in \mathcal{P}$, the map

$$x \to \omega(d, x)$$

is continuous. But observe that

$$\{x : \omega(d, x) = \frac{1}{2}\} = [-\hat{d}^+, \hat{0}^-] \cup [-\tilde{d}^+, \tilde{0}^-]$$

which is a clopen set. Thus, the set where the cocycle is 1 is also clopen, and so the cocycle is continuous.

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