Operator Algebras of Functions

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This is a joint work with Vern Paulsen

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Outline

- Quantized Function Theory: Overview

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- Goal of the talk

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Questions??

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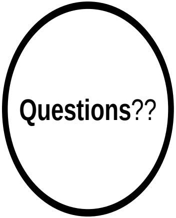

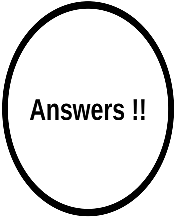

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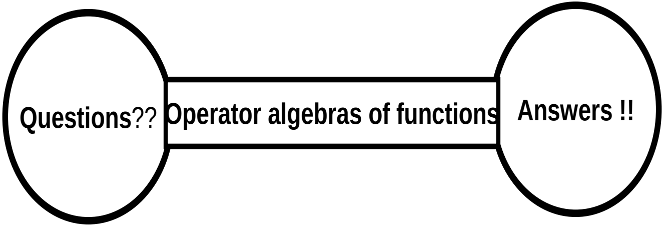

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- Examples

Overview: Quantized Function Theory

Domain:

$$G = \{z : \|F_k(z)\| < 1 \ \forall \ k \in I\} - \text{open subset of } C^N.$$

where I is some indexing set and $\mathcal{R} = \{F_k : G^- \to M_{m_k,n_k} : k \in I\}$ is a set of **analytic** functions.

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Quantized version of a domain:

$$\mathcal{Q}(G) = \{T : \sigma(T) \subseteq G \text{ and } \|F_k(T)\| \leq 1 \ \forall k \in I\}$$

where $T = (T_1, T_2, ..., T_N)$ is a commuting N-tuple of operators on some Hilbert space.

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Algebra:

$$H^\infty_{\mathcal{R}}(G) = \{f \in H^\infty(G) : \|f\|_{\mathcal{R}} < \infty\} \text{ where}$$

$\|f\|_{\mathcal{R}} = \sup\{\|f(T)\| : T \in \mathcal{Q}(G)\}$.

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An example of **"Abstract Operator Algebras of Functions."**

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The last question or rather slight variant of it has been partially answered. For finite indexing set I, it has been considered by Ambrozie-Timotin, Ball-Bolotnikov when $F_k's$ are matrix-valued polynomials.

Operator Algebras of Functions, General Theory

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Proposition

Let $\mathcal{A}$ be an operator algebra of functions on X, then $\mathcal{A} \subseteq \ell^\infty(X)$, and for every n and every $(f_{i,j}) \in M_n(\mathcal{A})$, we have $\|(f_{ij})\|_\infty \leq \|(f_{ij})\|_{M_n(\mathcal{A})}$ and $\|\pi_x\|_{cb} = 1$

General theory

Definition

A function $f : X \to \mathbb{C}$ *is called a BPW limit of* $\mathcal{A}$*, if there exists a net* $f_\lambda \in \mathcal{A}, f_\lambda \to f$ *ptw and* $\|f_\lambda\| \leq C$ *We let* $\tilde{\mathcal{A}}$ *denote the set of functions that are BPW limits from* $\mathcal{A}$*.*

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Theorem

If we equip $\tilde{\mathcal{A}}$ with the family of norms given by $\|(f_{ij})\| = inf\{C : \|(f_{ij}^\lambda)\|_{\mathcal{A}} \leq C,\ f_{ij}^\lambda \to f_{ij} \text{ ptw } \}$, then $\tilde{\mathcal{A}}$ is a BPW complete local operator algebra of functions.

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We call $\tilde{\mathcal{A}}$ the BPW completion of $\mathcal{A}$.

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2. $\tilde{\mathcal{A}}$ *is a dual operator algebra and if* (f_{ij}^{λ}) *is a bounded net in* $\mathcal{A}$, *then* $(f_{ij}^{\lambda}) \overset{wk*}{\rightarrow} (f_{ij}) \Leftrightarrow (f_{ij}^{\lambda}) \rightarrow (f_{ij})$ *ptw on X.*

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3. $\exists$ *a Hilbert space,* $\mathcal{H}$ *and* $\mathcal{H}$*-valued RKHS,* $\mathcal{L}$ *such that* $\tilde{\mathcal{A}} = \mathcal{M}(\mathcal{L})$ *complete isometric, wk*-isomorphism.*

Quantized Function Theory, Redux

Definition

Let $G \subseteq \mathbb{C}^N$ *be an open set. If there exists a set of matrix-valued functions,* $F_k = (f_{k,i,j}) : G^- \to M_{m_k,n_k}, k \in I$ *whose components are analytic functions on* G, *and* $G = \{z \in \mathbb{C}^N : \|F_k(z)\| < 1, k \in I\}$ *then we call* G *a* **analytically presented domain,**

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Definition

We say $\mathcal{R}$ is **separating analytic presentation of G** *if $\mathcal{A}$ separates points of G.*

Definition

The **quantized version of G** *is defined to be*

$$\mathcal{Q}(G) = \{T : \sigma(T) \subseteq G \text{ and } \|F_k(T)\| \leq 1, k \in I\},$$

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Definition

Let $\pi : \mathcal{A} \to B(\mathcal{H})$ *be the homomorphism of the algebra of the presentation. Then we call* π *an* **admissible representation** *provided that* $\|(\pi(f_{k,i,j}))\| \leq 1$ *for all* $k \in I$.

Definition

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Remark

Note that $\|.\|_{\mathcal{R}} \leq \|.\|_u$ *and consequently,* $\mathcal{A} \subseteq H^\infty_{\mathcal{R}}(G)$ *completely contractively.*

Theorem

Let G be an analytically presented domain with presentation $\mathcal{R} = \{F_k = (f_{k,i,j}) : G \to M_{m_k,n_k}, k \in I\}$, let $\mathcal{A}$ be the algebra of the presentation and let $P = (p_{ij}) \in M_{m,n}(\mathcal{A})$, where m, n are arbitrary. Then the following are equivalent:

(i) $\|P\|_u < 1$,

(ii) *there exists an integer l, matrices of scalars $C_j,\ 1 \leq j \leq l$ with $\|C_j\| < 1$ and admissible block diagonal matrices $D_j(z), 1 \leq j \leq l$, which are of compatible sizes and are such that*

$$P(z) = C_1 D_1(z) \cdots C_l D_l(z).$$

(iii) *there exists a positive, invertible matrix $R \in M_m$ and matrices $P_0, P_k \in M_{m,r_k}(\mathcal{A}), k \in K$, where $K \subseteq I$ is a finite set, such that*

$$I_m - P(z)P(w)^* = R + P_0(z)P_0(w)^* + \sum_{k \in K} P_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)} P_k(w)^*$$

where $r_k = q_k m_k$ and $z = (z_1, ..., z_N),\ w = (w_1, ..., w_N) \in G$.

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Let G be an analytically presented domain with a separating analytic presentation $\mathcal{R} = \{F_k : G \to M_{m_k,n_k} : k \in I\}$, let $\mathcal{A}$ be the algebra of the presentation and let $\widetilde{\mathcal{A}}$ be the BPW-completion of $\mathcal{A}$. Then

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2. *Norms the same if we use finite dimensional representations, since $H^\infty_{\mathcal{R}}(G)$ is RFD.*
3. $\exists$ $\mathcal{H}$*-valued RKHS, $\mathcal{L}$ such that $H^\infty_{\mathcal{R}}(G) = \mathcal{M}(\mathcal{L})$ complete isometric, wk*-isomorphism.*
4. $P = (p_{i,j}) \in M_{m,n}(H^\infty_{\mathcal{R}}(G))$ *and $\|P\|_{\mathcal{R}} \leq 1$ if and only if there exists K operator-valued analytic functions $R_k : G \to B(\mathcal{H}_k \otimes C^{m_k}, \mathbb{C}^m)$ such that*

$$I - P(z)P(w)^* = \sum_{k=1}^{K} R_k(z)[(I_m - F_k(z)F_k(w)^*) \otimes I_{\mathcal{H}_k}]R_k(w)^*.$$

when $I = \{1, \cdots, K\}$.

Goals accomplished

- Is $H^\infty_{\mathcal{R}}(G)$ a dual operator algebra? YES
- Is it possible to sup over only commuting matrices and achieve the same norm? YES
- Is $H^\infty_{\mathcal{R}}(G)$ a multiplier algebra of some RKHS? YES
- Is it possible to give a unified proof of Agler-type factorization result for $H^\infty_{\mathcal{R}}(G)$? YES

Goals accomplished

- Is $H^\infty_{\mathcal{R}}(G)$ a dual operator algebra? YES
- Is it possible to sup over only commuting matrices and achieve the same norm? YES but with some mild hypotheses.
- Is $H^\infty_{\mathcal{R}}(G)$ a multiplier algebra of some RKHS? YES
- Is it possible to give a unified proof of Agler-type factorization result for $H^\infty_{\mathcal{R}}(G)$? YES

Examples

1. **Polydisk:** $G = \mathbb{D}^N, I = \{1, \cdots, N\}$. For $1 \leq k \leq N$, define $F_k : G \to \mathbb{C}$ via $F_k(z) = z_k$, where $z = (z_1, \cdots, z_N)$

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In this case,

$$\|f\|_{\mathcal{R}} = \sup\{\|f(T)\| : T_i \in M_n \text{ and } T \in Q(\mathbb{D}^N)\}$$

where $T = (T_1, \cdots, T_N)$ is a commuting tuple of matrices.

Examples

1. Polydisk: $G = \mathbb{D}^N, I = \{1, \cdots, N\}$. For $1 \leq k \leq N$, define $F_k : G \to \mathbb{C}$ via $F_k(z) = z_k$, where $z = (z_1, \cdots, z_N)$ and

$$\mathcal{Q}(\mathbb{D}^N) = \{T : \sigma(T) \subseteq \mathbb{D}^N \text{ and } \|T_i\| \leq 1 \; \forall \; 1 \leq i \leq N\}.$$

The **algebra of the presentation** is the algebra of polynomials and clearly separates points of G.
$H^\infty_{\mathcal{R}}(G)$: Schur-Agler algebra.
In this case,

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4. Intersection of half planes: Let $G = \mathbb{H} = \{z \in C^N : Re(z_k) > 0 \ \forall \ k \in I\}$, where $z = (z_1, \cdots, z_N)$ and $I = \{1, \cdots, N\}$.

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These algebras have been studied by Kalyuzhnyi-Verbovetzkii.

5. Simply connected domain: Let G be a simply connected domain in $\mathbb{C}$, then $\exists$ a biholomorphic map $\phi : G \to \mathbb{D}$ such that

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$$\mathcal{R} = \{F_\theta(z) = \frac{z}{z - 2e^{i\theta}} : \ \theta \in I = [0, 2\pi)\}.$$

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And the list goes on ...