Noncommutative Semialgebraic sets

Multivariable operator theory workshop
Fields Institute

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Joint work with Tatiana Shulman

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Lifting against Star-Homomorphisms

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Theorem (Olsen 1971, Olsen-Pedersen 1989, Shulman 2008)

Suppose $\rho : B \to C$ is a surjective $*$-homomorphism between C^*-algebras. Given x in C with

$$\|x\| \leq 1 \text{ and } x^n = 0$$

there exists $\dot{x}$ in B with $\rho(\dot{x}) = x$ and

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C^*-Lifting, Ad Hoc Methods

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Relations	C^*-algebra	Liftable	Credit
$x^*x \leq 1$ $x^n = 0$	??	Yes	Olsen, Pedersen, Shulman
$x^*x \leq 1$ $\lVert x^n \rVert \leq \epsilon$	??	Yes	Olsen, Akemann, Pedersen
$x^*x = y^*y \leq 1$ $x^*y = y^*x = 0$	??	Yes	Loring, Pedersen
$x^*x = y^*y \leq 1$ $x^*y = y^*x = 0$ $x^2 = y^2 = 0$	$C_0\left((0,1], \mathbf{M}_3\right)$	Yes	Loring
$x^2 + y^2 \leq 1$ $x^* = x,\ y^* = y$ $xy = 0$	$C_0\left((0,1], \mathbb{C}^4\right)$	Yes	Loring
$x^*x = xx^* \leq 1$	$C_0\left(\mathbb{D} \setminus \{\mathbf{0}\}\right)$	No	Fredholm

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Semialgebraic means sets like

$$\left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \left|x^3 + y^2\right| \leq 1,\ |xy - x| \leq \epsilon \right\}$$

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and exact relations, unions, complements are also allowed.
contractibility will follow if we stick with homogeneous inequalities:

$$X=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; \left|x^2+y^2\right|\leq 1,\ \left|x^2y-xy^2\right|\leq\epsilon\right\}.$$

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there exists $\dot{x}$ and $\dot{y}$ in B with $\rho(\dot{x}) = x$, $\rho(\dot{y}) = y$, and

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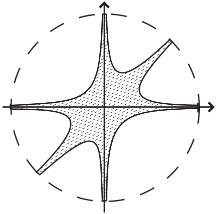
We could also discuss (normal) contractions with spectrum in X :



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Therefore

$$C_0(\mathbb{D} \setminus \{0\}) \cong \varinjlim C^* \left\langle x \,\middle|\, \|x^*x - xx^*\| \leq \frac{1}{n},\ \|x\| \leq 1 \right\rangle$$

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is a limit of projective C^*-algebras.

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A is a projective C^*-algebra if we can always solve the $*$-homomorphism lifting problem:

$$\begin{tikzcd} & B \arrow[d, "\rho"] \\ A \arrow[ur, dashed, "\exists\ \overline{\varphi}"] \arrow[r, "\varphi"'] & C \end{tikzcd}$$

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Theorem

If A is projective then A is residually finite dimensional.

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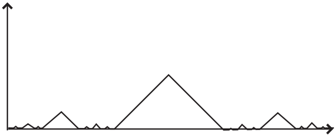

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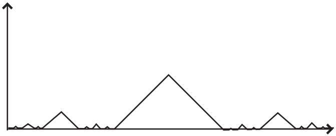

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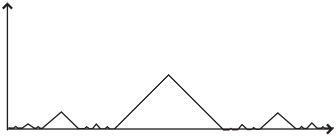

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We have

$$C_0(X \setminus \{0\}) \cong \varinjlim C^* \left\langle x \,\middle|\, 0 \le x \le 1,\ \|f(x)\| \le \frac{1}{3^n} \right\rangle$$

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To show that for any $\epsilon > 0$ the set of relations

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is liftable. A given lift is repeatedly improved using a quasicentral approximate unit.

Key Technical Lemma

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Lemma

Suppose A is a separable C^*-algebra and I is an ideal in A. Let π be the quotient map, and suppose u_n is a sequential quasi-central approximate unit for I relative to A. Then for a in A and $0 \leq \delta \leq 1$,

$$\limsup_n \|a(1-\delta u_n)\| \leq \max\left(\|\pi(a)\|, (1-\delta)\|a\|\right).$$

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Suppose $\rho : B \twoheadrightarrow C$ and

$$\|\rho(x)\| \leq 1 \text{ and } \|\rho(x)^*\rho(x) - \rho(x)\rho(x)^*\| \leq \epsilon.$$

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Even better will be $(1 - \delta_2 u_{n_2})(1 - u_{n_1})x$ and so on.

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Theorem

Suppose p is a NC $*$-polynomial in $x_1, \ldots, x_r$ that is homogeneous of degree greater than one and C is a positive constant. Then the universal C^*-algebra

$$C^* \left\langle x_1, \ldots, x_r \;\middle|\; \|p(x_1, \ldots, x_r)\| \leq \epsilon,\ \sum_{j=1}^{r} x_j x_j^* \leq C \right\rangle$$

is projective.

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is projective. (These relations are liftable.)

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Question

Are all the B_p isomorphic?