A collection of Browder joint spectra

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Overview

- Introduction
  - Browder essential spectrum
- Ascent and descent
  - Sets of operators
- Browder joint spectra
  - Spectral mapping theorem

Introduction: Browder operators

X Banach space
$B(X)$ bounded operators on X

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Equivalently,

- there exist closed a-invariant subspaces X_0, X_1 with $X = X_0 \oplus X_1$ such that X_0 is finite dimensional, $a|_{X_0}$ is nilpotent and $a|_{X_1}$ is invertible.
- $a = c + s$ where c is a compact operator, s is invertible and $cs = sc$.

Also known as Riesz-Schauder operators.
(Caradus, Pfaffenberger, Yood, 1977)

Introduction: Browder essential spectrum

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  acc = accumulation points
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- $\sigma_b(a) = \sigma(a) \setminus \{\text{Riesz points}\}$

  Riesz points = eigenvalues with finite multiplicity which are poles of the resolvent.

(F.E. Browder, 1961)

Introduction: Browder essential spectrum

Spectral mapping theorem: (Gramsch and Lay, 1972)
If a function $f(z)$ is analytic on a neighbourhood of the spectrum $\sigma(a)$ then

$$\sigma_b(f(a)) = f(\sigma_b(a))$$

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Let $a \in B(X)$.

Proposition

Denote by $\mathcal{C}(a)$ the set of compact operators c with $ac = ca$. Then

$$\sigma_b(a) = \bigcap_{c \in \mathcal{C}(a)} \sigma(a + c)$$

(D.C. Lay, 1968)

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Proposition

Denote by $\mathcal{Q}(a)$ the set of finite rank projections $q \in B(X)$ with $aq = qa$.

$$\sigma_b(a) = \bigcap_{q \in \mathcal{Q}(a)} \sigma(a|_{(I-q)X})$$

(J. Zemánek, 1986)

Introduction

Question: To what extent can the single variable theory of the Browder essential spectrum be extended to a multivariable setting?

Introduction: Browder joint spectra

There are a number of different types of Browder joint spectrum in the literature...

- M. Snow (1975)
  - bicommutant Browder joint spectrum
- J.J. Buoni, A.T. Dash, B.L. Wadhwa (1981)
  - polynomial Browder joint spectrum
- R.E. Curto, A.T. Dash (1988)
  - Browder spectral systems
- V. Kordula, V. Müller, V. Rakočević (1997)
  - semi-Browder spectra

Introduction

We will construct a new Browder joint spectrum by extending the notions of ascent and descent for an operator.

Ascent and Descent: single operator case

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If a has finite ascent and finite descent then they must be equal.

Ascent and Descent: single operator case

Let $a \in B(X)$.
The following are equivalent:

(i) a has finite ascent and descent with $r = \alpha(a) = \delta(a)$

(ii) $X = \text{Ker } a^r \oplus \text{Ran } a^r$ but $X \neq \text{Ker } a^{r-1} \oplus \text{Ran } a^{r-1}$

(iii) $0 \in \mathbb{C}$ is a pole of the resolvent of a of order r

(iv) a has a Drazin inverse with index r (i.e. there exists $d \in B(X)$ such that $ad = da$, $d = ad^2$ and $a^r = a^{r+1}d$ but $a^{r-1} \neq a^r d$)

Brief history of ascent and descent

- F. Riesz (1916) - compact operators
- A.F. Ruston (1954) - Riesz operators
- H. Heuser (1956) - ascent, descent, nullity and defect
- A.E. Taylor (1958/66) - poles of the resolvent
- S. Grabiner (1978) - compact perturbations
- Mbekhta, Müller (1996) - essential ascent and descent
- Grunenfelder, Omladič (1999) - commuting module endomorphisms

Ascent and Descent: sets of operators

For each set $A \subseteq L(X)$ of operators define

$$N(A) = \bigcap_{a \in A} \operatorname{Ker} a \qquad R(A) = \operatorname{span}\Big(\bigcup_{a \in A} \operatorname{Ran} a\Big).$$

$$A^k = \{a_1 \dots a_k : a_1, \dots, a_k \in A\} \qquad A^0 = \{I\}$$

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The ascending chain length of A, denoted $\text{acl}(A)$, is the length of the chain

$$\{0\} \subseteq N(A) \subseteq N(A^2) \subseteq N(A^3) \subseteq N(A^4) \subseteq N(A^5) \subseteq \cdots$$

The descending chain length of A, denoted $\text{dcl}(A)$, is the length of the chain

$$X \supseteq R(A) \supseteq R(A^2) \supseteq R(A^3) \supseteq R(A^4) \supseteq R(A^5) \supseteq \cdots$$

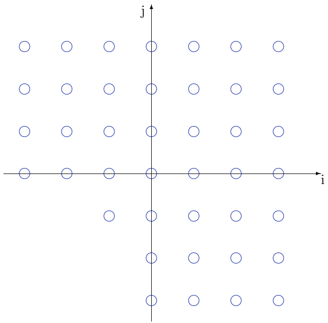
Chain lengths are not enough to extend single variable theory...

Example

$H =$ Hilbert space with orthonormal basis $(e_{i,j})$ indexed by $(\mathbb{Z}^+ \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{Z}^+) \cup \{(-1,-1)\}$.
$A = \{a_1, a_2\}$ where $a_1(e_{i,j}) = e_{i+1,j}$ and $a_2(e_{i,j}) = e_{i,j+1}$.

- a_1 is a right shift
- a_2 is an upward shift
- ascending chain length of A is 0
- descending chain length of A is 1

No decomposition of space.



Alternative description of ascent and descent

In the case of a single operator a there are isomorphisms

$$\text{Ker } a^{r+1}/\text{Ker } a^r \cong \text{Ker } a \cap \text{Ran } a^r$$

$$\text{Ran } a^r/\text{Ran } a^{r+1} \cong X/(\text{Ker } a^r + \text{Ran } a)$$

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So the ascent of a is the smallest r such that

$$\text{Ker } a \cap \text{Ran } a^r = \{0\}$$

and the descent of a is the smallest r such that

$$\text{Ker } a^r + \text{Ran } a = X.$$

Ascent and descent for sets of operators

This motivates the following definition for a set A of operators.

The ascent of A, denoted $\alpha(A)$, is the smallest r such that

$$N(A) \cap R(A^r) = \{0\}$$

The descent of A, denoted $\delta(A)$, is the smallest r such that

$$N(A^r) + R(A) = X$$

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Proposition

(i) *$acl(A) \leq \alpha(A)$ and $dcl(A) \leq \delta(A)$*

(ii) *if $\alpha(A) < \infty$ then $\alpha(A) \leq dcl(A)$*

(iii) *if $\delta(A) < \infty$ then $\delta(A) \leq acl(A)$*

(iv) *if $\alpha(A), \delta(A) < \infty$ then $\alpha(A) = acl(A) = dcl(A) = \delta(A)$*

Main Decomposition Theorem

Let A be a set of operators on a vector space X.

Theorem

A has finite ascent and finite descent if and only if there exist A-invariant subspaces X_1, X_2 of X such that

(i) $X = X_1 \oplus X_2$,

(ii) $A^k|_{X_1} = \{0\}$ *some k (nilpotent),*

(iii) $N(A|_{X_2}) = 0$ *and* $R(A|_{X_2}) = X_2$ *(bijective).*

Moreover, the spaces X_1 and X_2 are given uniquely by $X_1 = N(A^r)$ and $X_2 = R(A^r)$, where r is the common value of the ascent and descent of A.

Some remarks

$A \subset B(X)$

- Let $B \subseteq \text{commutant}(A)$. If A and B both have finite ascent and finite descent then $A \cup B$ has finite ascent and finite descent.

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  (i) $acl(A) = acl(\langle A \rangle)$ and $dcl(A) = dcl(\langle A \rangle)$,
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- If $R(A^k)$ is closed for all k then
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- If A is countable with finite ascent and finite descent then
  (i) $R(A^r)$ is closed where $r = \alpha(A) = \delta(A)$
  (ii) the closed algebra generated by A has finite ascent and descent r

Ascent and descent for tuples of operators

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-tuple of operators on X.

We write...

$$\alpha(\mathbf{a}) = \alpha(A) \qquad \delta(\mathbf{a}) = \delta(A) \qquad \text{where } A = \bigcup_{j=1}^{n} \{a_j\}.$$

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$\mathbf{a}^k = (a_{i_1} \ldots a_{i_k})_{i_1,\ldots,i_k=1}^{n}$ (lexicographic order)

$\mathrm{row}(\mathbf{a}) : X^n \to X \qquad (x_1, \ldots, x_n) \mapsto a_1 x_1 + \cdots + a_n x_n$

$\mathrm{col}(\mathbf{a}) : X \to X^n \qquad x \mapsto (a_1 x, \ldots, a_n x)$

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Proposition

(i) $\alpha(\mathbf{a}) = \min\{r \in \mathbb{N}_0 : \mathit{Ker\ row}(\mathbf{a}^r) = \mathit{Ker\ col}(\mathbf{a}) \circ \mathit{row}(\mathbf{a}^r)\}$

(ii) $\delta(\mathbf{a}) = \min\{r \in \mathbb{N}_0 : \mathit{Ran\ col}(\mathbf{a}^r) = \mathit{Ran\ col}(\mathbf{a}^r) \circ \mathit{row}(\mathbf{a})\}$

Remarks for *m*-tuples of operators

Let $\mathbf{a} = (a_1, \ldots, a_m)$ be an m-tuple of operators on X.

- If $n(\mathbf{a}) = \dim N(\mathbf{a})$ and $d(\mathbf{a}) = \operatorname{codim} R(\mathbf{a})$ then
  - (i) $n(\mathbf{a}^k) \leq (1 + m + \cdots + m^{k-1})n(\mathbf{a})$ for all $k \in \mathbb{N}$,
  - (ii) $d(\mathbf{a}^k) \leq (1 + m + \cdots + m^{k-1})d(\mathbf{a})$ for all $k \in \mathbb{N}$,
  - (iii) If $n(\mathbf{a}) < \infty$ and $\delta(\mathbf{a}) < \infty$ then $d(\mathbf{a}) < \infty$,
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- If $\mathbf{a}' = (a'_1, \ldots, a'_m)$ is the m-tuple of transpose operators acting on the algebraic conjugate X' then
  (i) $\mathrm{acl}(\mathbf{a}) = \mathrm{dcl}(\mathbf{a}')$ and $\mathrm{dcl}(\mathbf{a}) = \mathrm{acl}(\mathbf{a}')$
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This last result does not hold for non-commuting tuples: eg. if $\mathbf{a} = (a_1, a_2)$ where $a_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\delta(\mathbf{a}) = 0$ and $\alpha(\mathbf{a}) = \infty$.

Fredholm Tuples

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a tuple of bounded operators on a Banach space X.

$\mathbf{a}$ is called upper semi-Fredholm if $\mathrm{col}(\mathbf{a})$ is upper semi-Fredholm.
$\mathbf{a}$ is called lower semi-Fredholm if $\mathrm{row}(\mathbf{a})$ is lower semi-Fredholm.
$\mathbf{a}$ is called Fredholm if it is both upper and lower semi-Fredholm.

Fredholm Tuples

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a tuple of bounded operators on a Banach space X.

$\mathbf{a}$ is called upper semi-Fredholm if $\mathrm{col}(\mathbf{a})$ is upper semi-Fredholm.
$\mathbf{a}$ is called lower semi-Fredholm if $\mathrm{row}(\mathbf{a})$ is lower semi-Fredholm.
$\mathbf{a}$ is called Fredholm if it is both upper and lower semi-Fredholm.

The upper Fredholm spectrum of $\mathbf{a}$ is

$$\sigma_{\pi e}(\mathbf{a}) = \{\lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \text{ is not upper semi-Fredholm}\}.$$

The lower Fredholm spectrum of $\mathbf{a}$ is

$$\sigma_{\delta e}(\mathbf{a}) = \{\lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \text{ is not lower semi-Fredholm}\}.$$

(J. Buoni, R. Harte, T. Wickstead, 1977)

A Browder joint spectrum

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a commuting tuple of bounded operators on a Banach space X.

Definition

$\mathbf{a}$ is called a Browder tuple on X if $\mathbf{a}$ is Fredholm with finite ascent and finite descent.

The Browder joint spectrum of $\mathbf{a}$ is

$$\sigma_b(\mathbf{a}) = \{\lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \text{ is not Browder}\}.$$

Properties of Browder joint spectrum

Proposition

If X is infinite dimensional then $\sigma_b(\mathbf{a})$ is non-empty and compact.
If X is finite dimensional then $\sigma_b(\mathbf{a}) = \emptyset$.

Inclusions (which can be strict):

$$\sigma_{B-}(\mathbf{a}) \cup \sigma_{B+}(\mathbf{a}) \quad \subseteq \quad \sigma_b(\mathbf{a}) \quad \subseteq \quad \sigma_{Tb}(\mathbf{a})$$

(Kordula, Müller, Rakočević) (Curto, Dash)

Adjoints

Let $\mathbf{a} = (a_1, \dots, a_n)$ be a tuple of bounded operators on a Banach space X and let $\mathbf{a}^* = (a_1^*, \dots, a_n^*)$ be the tuple of adjoint operators on the dual space X^*.

Proposition

If $\mathbf{a} = (a_1, \dots, a_n)$ is Fredholm then

(i) *$acl(\mathbf{a}) = dcl(\mathbf{a}^*)$ and $dcl(\mathbf{a}) = acl(\mathbf{a}^*)$,*

(ii) *$\alpha(\mathbf{a}) = \delta(\mathbf{a}^*)$ and $\delta(\mathbf{a}) = \alpha(\mathbf{a}^*)$.*

Corollary

$\sigma_b(\mathbf{a}) = \sigma_b(\mathbf{a}^*)$.

Projection Property

Proposition

$\sigma_b(\mathbf{a})$ satisfies the projection property.

i.e. for all coordinate projections $p : \mathbb{C}^n \to \mathbb{C}^k$ we have

$$p(\sigma_b(\mathbf{a})) = \sigma_b(p(\mathbf{a})).$$

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Proof

Let $p : \mathbb{C}^n \to \mathbb{C}^k$ be the projection $p(z_1, \ldots, z_n) = (z_{i_1}, \ldots, z_{i_k})$.
Write $p(\mathbf{a}) = (a_{i_1}, \ldots, a_{i_k})$.
We will show $p(\sigma_b(\mathbf{a})) \supseteq \sigma_b(p(\mathbf{a}))$.

Suppose $0 \in \sigma_b(p(\mathbf{a}))$. Need to find $\lambda \in \sigma_b(\mathbf{a})$ with $p(\lambda) = 0$.

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Three cases to consider...

(i) $p(\mathbf{a})$ is not Fredholm

(ii) $p(\mathbf{a})$ is Fredholm but has infinite ascent

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(i) If $p(\mathbf{a})$ is not upper (lower) semi-Fredholm then we can use the projection property for the upper (lower) Fredholm spectrum to find $\lambda \in \sigma_b(\mathbf{a})$ with $p(\lambda) = 0$.

(ii) Suppose $p(\mathbf{a})$ is Fredholm with infinite ascent.
Then for each r, $Y^{(r)} = N(p(\mathbf{a})) \cap R(p(\mathbf{a})^r)$ is non-zero and finite dimensional.

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Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ where $p(\lambda) = 0$ and $q(\lambda) = \mu$.
Then $N(\mathbf{a} - \lambda) \cap R((\mathbf{a} - \lambda)^r)$ is non-zero for all r.
Hence $\mathbf{a} - \lambda$ has infinite ascent and so $\lambda \in \sigma_b(\mathbf{a})$.

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(iii) Suppose $p(\mathbf{a})$ is Fredholm with infinite descent.
Then $p(\mathbf{a})^*$ is Fredholm with infinite ascent.
Use argument in (ii) to find $\lambda \in \sigma_b(\mathbf{a}^*) = \sigma_b(\mathbf{a})$ such that $p(\lambda) = 0$. $\square$

Characterisation of Browder spectrum

Let $\mathbf{a} = (a_1, \dots, a_n)$ be a commuting n-tuple of bounded operators on X.
Denote by $\mathcal{C}_n(\mathbf{a})$ the set of commuting n-tuples $\mathbf{c} = (c_1, \dots, c_n)$ of compact operators such that $a_i c_j = c_j a_i$ for each i, j.

Proposition

$$\sigma_b(\mathbf{a}) = \bigcap_{\mathbf{c} \in \mathcal{C}_n(\mathbf{a})} \{\sigma_\pi(\mathbf{a} + \mathbf{c}) \cup \sigma_\delta(\mathbf{a} + \mathbf{c})\}$$

σ_π = joint approximate point spectrum
σ_δ = joint defect spectrum

(D.C. Lay, n=1, 1968)

Characterisation of Browder spectrum

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-tuple of bounded operators on X. Denote by $\mathcal{Q}(\mathbf{a})$ the set of finite rank projections $q \in B(X)$ with such that $a_j q = q a_j$ for each $j = 1, \ldots, n$.

Proposition

$$\sigma_b(\mathbf{a}) = \bigcap_{q \in \mathcal{Q}(\mathbf{a})} \{\sigma_\pi(\mathbf{a}|_{(I-q)X}) \cup \sigma_\delta(\mathbf{a}|_{(I-q)X})\}$$

σ_π = joint approximate point spectrum
σ_δ = joint defect spectrum

(J. Zemánek, n=1, 1986)

Characterisation of Browder Tuples

The following are equivalent:

- $\mathbf{a} = (a_1, \ldots, a_n)$ is a Browder tuple
- there exist $\mathbf{a}$-invariant subspaces X_1, X_2 of X such that
  (i) $X = X_1 \oplus X_2$ where X_1 is finite dimensional and X_2 is closed,
  (ii) $\mathbf{a}|_{X_1}$ is nilpotent,
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- $\mathbf{a} = \mathbf{c} + \mathbf{s}$ where
  (i) $\mathbf{c} = (c_1, \ldots, c_n)$ are commuting compact operators
  (ii) $\mathbf{s} = (s_1, \ldots, s_n)$ is a commuting n-tuple which is jointly bounded below and jointly onto
  (iii) $c_i s_j = s_j c_i$ for all i, j

Extension of the spectral mapping theorem

This Browder joint spectrum concerns the first and last stages of the Koszul complex

$$0 \to \Lambda^0[X] \xrightarrow{\delta^0_{\mathbf{a}}} \Lambda^1[X] \xrightarrow{\delta^1_{\mathbf{a}}} \cdots \xrightarrow{\delta^{n-2}_{\mathbf{a}}} \Lambda^{n-1}[X] \xrightarrow{\delta^{n-1}_{\mathbf{a}}} \Lambda^n[X] \to 0$$

Let $\tilde{\sigma}$ denote any of the Slodkowski spectra which involve the first and last stages (i.e. $\sigma_{\pi,k} \cup \sigma_{\delta,l}$). Then a spectral mapping theorem holds for

$$\tilde{\sigma}_b(\mathbf{a}) = \bigcap_{\mathbf{c} \in \mathcal{C}_n(\mathbf{a})} \tilde{\sigma}(\mathbf{a} + \mathbf{c})$$

where $\mathcal{C}_n(\mathbf{a})$ is the set of commuting n-tuples $\mathbf{c} = (c_1, \ldots, c_n)$ of compact operators such that $a_i c_j = c_j a_i$ for each i, j.

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This result fails for the one-sided Slodkowski spectra...

Example

$H =$ Hilbert space with orthonormal basis $(e_{i,j})$ indexed by $(\mathbb{N}_0 \times \mathbb{N}_0) \cup \{(-1,0)\}$.

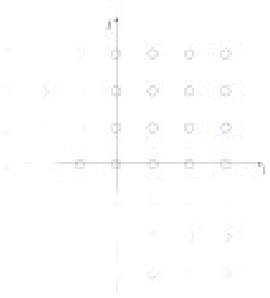
$$\mathbf{a} = (a_1, a_2) \text{ where } a_1(e_{i,j}) = \begin{cases} e_{i,j-1} & \text{if } j > 0 \\ 0 & \text{if } j = 0 \end{cases}$$

$$\text{and } a_2(e_{i,j}) = e_{i+1,j}.$$

Consider the defect spectrum σ_δ.

- a_1 is a downward shift
- a_2 is an right shift
- $(0,0) \in \sigma_{\delta,b}(\mathbf{a})$
- $0 \notin \sigma_{\delta,b}(a_1)$

No spectral mapping theorem in this case.



Taylor-Browder spectrum

We recover the Taylor-Browder spectrum of Curto and Dash with the formula

$$\sigma_{Tb}(\mathbf{a}) = \bigcap_{\mathbf{c} \in \mathcal{C}_n(\mathbf{a})} \sigma_T(\mathbf{a} + \mathbf{c})$$

The following are equivalent:

- $\mathbf{a} = (a_1, \ldots, a_n)$ is Taylor-Browder,
- $0 \in \mathbb{C}^n$ is not an accumulation point in $\sigma_T(\mathbf{a})$ and $\mathbf{a}$ has finite ascent, descent and nullity,
- $\mathbf{a} = \mathbf{c} + \mathbf{s}$ where
  - (i) $\mathbf{c} = (c_1, \ldots, c_n)$ are commuting compact operators
  - (ii) $\mathbf{s} = (s_1, \ldots, s_n)$ is a commuting n-tuple which is Taylor invertible
  - (iii) $c_i s_j = s_j c_i$ for all i, j

Articles

- D. Kitson, *Ascent and descent for sets of operators.* (Studia Math., 2009)
- R.E. Harte, D. Kitson, *On Browder tuples.* (Acta Sci. Math. (Szeged), 2009)

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Thank you