Wandering vectors and the (hyper-)reflexivity of free semigroup algebras

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Free semigroup algebras

Definition
A *free semigroup algebra* is the weakly closed (non-self-adjoint) algebra generated by n isometries $S_1, ..., S_n$ with pairwise orthogonal ranges, i.e. such that
$$S_i^* S_j = \delta_{ij} I.$$

.

The n-tuple $[S_1, ..., S_n]$ is called a *row isometry*, since the row operator $[S_1 \cdots S_n]$ is an isometry from $\mathcal{H}^n$ to $\mathcal{H}$.

Non-commutative dilation theory

Definition
A tuple $[A_1, A_2, ...]$ is called a *row contraction* if the row operator $[A_1 A_2 \cdots]$ is contractive, or equivalently, if $\sum A_i A_i^* \leq I$.

Theorem (Frahzo, Bunce, Popescu)
A row contraction $A = [A_1 A_2 \cdots]$ *dilates to a row isometry* $[S_1 S_2 \cdots]$, *where*

$$S_i = \begin{pmatrix} A_i & 0 \\ * & * \end{pmatrix}.$$

The non-commutative analytic Toeplitz algebra

Definition (Popescu)

Let $\mathbb{F}_n^+$ denote the free semigroup algebra in n non-commuting letters $\{1, ..., n\}$, and let $\mathcal{F}$ denote the "Fock" space $\ell^2(\mathbb{F}_n^+)$ with orthonormal basis $\{\xi_w : w \in \mathbb{F}_n^+\}$ consisting of words in $\mathbb{F}_n^+$.

Define n isometries $L_1, ..., L_n$ on $\mathcal{F}$ by

$$L_i \xi_w = \xi_{iw}, \qquad w \in \mathbb{F}_n^+.$$

These isometries have pairwise orthogonal ranges. The free semigroup algebra they generate is the *non-commutative analytic Toeplitz algebra* (sometimes called the *left regular representation algebra)*, which we denote by $\mathcal{L}_n$.

The non-commutative analytic Toeplitz algebra (2)

When $n = 1$, $\mathcal{L}_n$ is the familiar analytic Toeplitz algebra generated by the unilateral shift. For $n = 2$ we have the following diagram:

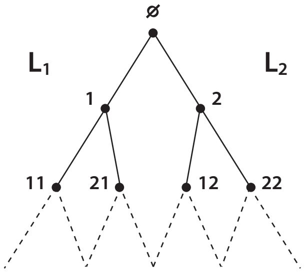


Figure:

For a word $w = w_1 \cdots w_k$ in $\mathbb{F}_n^+$, write $L_w = L_{w_1} \cdots L_{w_k}$. An element A of $\mathcal{L}_n$ is completely determined by its Fourier series

$$A \sim \sum_{w \in \mathbb{F}_n^+} a_w L_w,$$

the Cesaro sums of which converge strongly to A.

Wandering spaces and Beurling's Theorem

Definition

A subspace $\mathcal{W}$ is said to be *wandering* for a free semigroup algebra $\mathcal{S}$ if the subspaces $S_w \mathcal{W}$ are pairwise orthogonal for distinct words in $\mathbb{F}_n^+$. A vector w is said to be *wandering* if the subspace $\mathbb{C}w$ is wandering.

For $n = 1$ we have the following classical result.

Theorem (Beurling's Theorem)

Every invariant subspace of the unilateral shift is generated by a wandering vector.

For $n \geq 2$ we have the following generalization.

Theorem (Popescu)

Every invariant subspace of $\mathcal{L}_n$ is a direct sum of cyclic subspaces generated by wandering vectors.

The right regular representation algebra

Definition

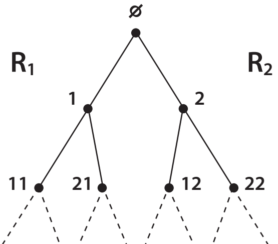
Once again, let $\mathcal{F}$ denote the "Fock" space $\ell^2(\mathbb{F}_n^+)$, and define n isometries $R_1, ..., R_n$ on $\mathcal{F}$ by

$$R_i \xi_w = \xi_{wi}, \qquad w \in \mathbb{F}_n^+.$$

The free semigroup algebra generated by these isometries is the *right regular representation algebra*, which we denote by $\mathcal{R}_n$.

Theorem (Davidson, Pitts)

The free semigroup algebra $\mathcal{R}_n$ is unitarily equivalent to $\mathcal{L}_n$, and is the commutant of $\mathcal{L}_n$.



The structure of free semigroup algebras

Definition

For a free semigroup algebra $\mathcal{S}$, let $\mathcal{S}_0$ denote the weakly closed ideal generated by $S_1, ..., S_n$.

Note

Either $\mathcal{S}_0 = \mathcal{S}$ or $\mathcal{S}/\mathcal{S}_0 \cong \mathbb{C}$.

Theorem (Dichotomy)

We have the following dichotomy:

1. *If $\mathcal{S}_0 = \mathcal{S}$, then $\mathcal{S}$ is a Von Neumann algebra.*
2. *Otherwise, $\mathcal{S}_0$ is a proper idea, and there is a unique weakly continuous homomorphism Φ of $\mathcal{S}$ onto $\mathcal{L}_n$ such that $\Phi(S_i) = L_i$. In this case, $\mathcal{S}/\ker\Phi \cong \mathcal{L}_n$.*

The first case can occur:

Example (Read)

$\mathcal{B}(\mathcal{H})$ is a free semigroup algebra.

The structure of free semigroup algebras (2)

Definition

A free semigroup algebra is said to be of *type L* if it is (algebraically) isomorphic to $\mathcal{L}_n$.

Note

If $\mathcal{S}$ has a wandering vector w then the restriction of $\mathcal{S}$ to $\mathcal{S}[w]$ is isomorphic (in fact, unitarily equivalent) to $\mathcal{L}_n$.

Theorem (Davidson, Katsoulis, Pitts)

If $\mathcal{S}$ is of type L then there is a canonical map $\Phi : \mathcal{S} \to \mathcal{L}_n$ taking $\Phi(S_i) = L_i$ which is a completely isometric isomorphism and a weak-weak* homeomorphism.*

If $\mathcal{S}$ is non-self-adjoint, then it has a type L part:

Theorem (DKP)

Let $\mathcal{S}$ be a free semigroup algebra. Then $\mathcal{S}$ contains a largest projection P such that $P\mathcal{S}P$ is self-adjoint, the range of $P^\perp$ is invariant for $\mathcal{S}$, and $\mathcal{S}P^\perp (= P^\perp \mathcal{S}P)$ is type L.

Wandering vectors and reflexivity

Definition

An operator algebra $\mathcal{A}$ is *reflexive* if it can be recovered from its invariant subspace lattice $\mathsf{Lat}(\mathcal{A})$ as the set of all operators $\mathsf{Alg}(\mathsf{Lat}(\mathcal{A}))$ leaving each subspace invariant.

Theorem (DKP)

A free semigroup algebra is reflexive if and only if its type L part is reflexive.

Theorem (Davidson, Li, Pitts)

A type L free semigroup algebra is reflexive if it has a wandering vector.

Theorem (Davidson, Kribs, Shpigel)

A free semigroup algebra without a wandering vector is reductive. Hence a type L free semigroup algebra is reflexive if and only if it has a wandering vector.

Question (DKP)

Is every free semigroup algebra reflexive? Equivalently, does every type L free semigroup algebra have a wandering vector?

Hyper-reflexivity

Definition (Hyper-reflexivity)

An operator algebra $\mathcal{A}$ *hyper-reflexive* if the semi-norms

$$\beta(T) = \sup_{\mathsf{Lat}(\mathcal{A})} \|P^{\perp} TP\|$$

and $\mathsf{dist}(T, \mathsf{Alg}(\mathcal{A}))$ are comparable. In this case, the *hyper-reflexivity constant* is the smallest C such that

$$\beta(T) \leq C\, \mathsf{dist}(T, \mathsf{Alg}(\mathcal{A})), \qquad T \in \mathcal{B}(\mathcal{H}).$$

Note

It is always the case that

$$\mathsf{dist}(T, \mathsf{Alg}(\mathcal{A})) \leq \beta(T),$$

and $\beta(T) = 0$ precisely when T belongs to $\mathsf{Alg}(\mathcal{A})$. If $\mathcal{A}$ is hyper-reflexive, then it is reflexive, but the converse is not true.

Hyper-reflexivity (2)

Theorem (Davidson)

The analytic Toeplitz algebra is hyper-reflexive with constant 19.

Theorem (DP)

For $n \geq 2$, $\mathcal{L}_n$ *is hyper-reflexive with constant* 51.

This was later improved:

Theorem (Bercovici)

For $n \geq 2$, $\mathcal{L}_n$ *is hyper-reflexive with constant* 3.

Theorem (DLP)

If $\mathcal{S}$ *is a type L free semigroup algebra with a wandering vector, then* $\mathcal{S}$ *is hyper-reflexive with constant* 55.

Dual algebras techniques

Definition

A *dual algebra* is a unital weak*-closed subalgebra of $\mathcal{B}(\mathcal{H})$, the predual of which can be identified with a quotient of the trace class operators.

The term “dual algebra techniques” refers to a collection of ideas which have been used to, among other things, prove the existence of the invariant subspaces of certain dual algebras. Roughly, the idea is to show that the predual of the algebra is small in a certain sense.

First introduced by Scott Brown (and often referred to as the “Scott Brown Technique”):

Theorem (Brown)

Subnormal operators have invariant subspaces.

Dual algebra techniques have been utilized, with a great deal of success, in both single operator theory and the theory of commuting contractions.

Dual algebra techniques (2)

Definition

Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{B}(\mathcal{H})$. Then $\mathcal{A}$ is said to have property $\mathbb{A}_1$ ($\mathbb{A}_1(1)$) if, for every weak*-continuous linear functional φ on $\mathcal{A}$ and $\epsilon > 0$, there are vectors x and y in $\mathcal{H}$ (with $\|x\|\|y\| < (1+\epsilon)\|\varphi\|$) such that

$$\varphi(A) = (Ax, y) \qquad A \in \mathcal{A}.$$

Theorem (Davidson, Katsoulis, Pitts)

$\mathcal{L}_n$ has property $\mathbb{A}_1(1)$.

Corollary

The weak and weak topologies agree on $\mathcal{L}_n$.*

Dual algebra techniques for free semigroup algebras

Setup

Let $\mathcal{S}$ be a free semigroup algebra which is non-self-adjoint (so that $I \notin \mathcal{S}_0$).

Idea

Suppose $\mathcal{S}$ has property $\mathbb{A}_1$. Let φ be the weak*-continuous linear functional which annihilates $\mathcal{S}_0$ and satisfies $\varphi(I) = 1$. Then we can find vectors x and y (necessarily nonzero) such that

$$\varphi(A) = (Ax, y), \qquad A \in \mathcal{S}.$$

Conclusion (Wandering Vector)

Since $\mathcal{S}_0[x]$ is orthogonal to y, it is a proper invariant subspace for $\mathcal{S}$. But $\mathcal{S}[x] \ominus \mathcal{S}_0[x]$ is nonzero since $(x, y) = 1$. Any unit vector z in $\mathcal{S}[x] \ominus \mathcal{S}_0[x]$ is a wandering vector, since $\mathcal{S}_0[z] \subseteq \mathcal{S}_0[x]$ means $\mathcal{S}_0[z]$ is orthogonal to z, giving

$$(S_u z, S_v z) = (S_v^* S_u z, z) = \delta_{uv}.$$

Dual algebra techniques for free semigroup algebras (2)

Goal

Prove that every type L free semigroup algebra $\mathcal{S}$ has property $\mathbb{A}_1$.

Definition

For vectors x and y, let $[x \otimes y]$ denote the linear functional mapping $A \to (Ax, y)$.

The following technical "trick" is often employed to show that a dual algebra has property $\mathbb{A}_1$.

Goal

Let φ be a weak*-continuous linear functional on $\mathcal{S}$. We want to find convergent sequences (x_k) and (y_k) such that

$$\|\varphi - [x_k \otimes y_k]\| \to 0,$$

since this will give $\varphi = [x \otimes y]$, where $x = \lim x_k$ and $y = \lim y_k$.

Dual algebra techniques for free semigroup algebras (3)

Idea

Fix x_k and y_k. Suppose we can find vectors u and v such that

1. $[u \otimes v]$ approximates the error $\varphi - [x_k \otimes y_k]$ arbitrarily closely,
2. $\|[x_k \otimes v]\|$ and $\|[u \otimes y_k]\|$ are arbitrarily small, and
3. $\|u\|$ and $\|v\|$ are arbitrarily close to the size of the error $\|\varphi - [x_k \otimes y_k]\|$.

Set $x_{k+1} = x - u$, $y_{k+1} = y_k - v$. Then

$$\|\varphi - [x_{k+1} \otimes y_{k+1}]\| \leq \|\varphi - [x_k \otimes y_k] - [u \otimes v]\| + \|[x_k \otimes v]\| + \|[u \otimes y_k]\|,$$

so $[x_{k+1} \otimes y_{k+1}]$ is a better approximation to φ, and the sequences (x_n) and (y_n) can be made Cauchy.

Intertwining operators

In the context of a commutative algebra, arguments of this type typically rely on function and measure-theoretic tools which are not obviously available in our non-commutative setting. We rely instead on certain intertwining operators.

Definition

Let $\mathcal{S}$ be a free semigroup algebra on $\mathcal{H}$. An operator $X : \mathcal{F} \to \mathcal{H}$ is said to *intertwine* $\mathcal{L}_n$ and $\mathcal{S}$ if

$$XL_w = XL_w, \qquad w \in \mathbb{F}_n^+.$$

Theorem (DKP)

If $\mathcal{S}$ is type L, then every vector in $\mathcal{H}$ is in the range of such an intertwining operator.

Intertwining operators (2)

Theorem (DKP)

If $\mathcal{S}$ is type L, then for some $m \geq 1$, the ampliation $\mathcal{S}^{(m)}$ has a wandering vector.

Construction

Suppose $m \geq 1$, $\mathcal{S}^{(m)}$ has a wandering vector $\overline{w} = (w_1, ..., w_m)$. The restriction of $\mathcal{S}^{(m)}$ to $\mathcal{S}^{(m)}[\overline{w}]$ is unitarily equivalent to $\mathcal{L}_n$. Compose this equivalence with a projection to get a bounded operator $X : \mathcal{F} \to \mathcal{H}$ satisfying

$$L_w \xi_0 \to S_w w_1, \qquad w \in \mathbb{F}_n^+.$$

The operator X intertwines $\mathcal{S}$ and $\mathcal{L}_n$.

L-Toeplitz operators

Recall that the Toeplitz operators are precisely the operators T satisfying $S^*TS = T$, where S is the unilateral shift. This motivates:

Definition

An operator T is said to be an *L-Toeplitz Operator* if

$$R_i^* T R_i = T \quad \text{for } 1 \leq i \leq n.$$

The L-Toeplitz operators are precisely the weak-operator closure of the operator system $\mathcal{L}_n + \mathcal{L}_n^*$. Popescu showed that every L-Toeplitz operator T is completely determined by its Fourier series

$$T \sim \sum_{w \in \mathbb{F}_n^+} a_w L_w + \sum_{w \in \mathbb{F}_n^+ \setminus \emptyset} \overline{b_w} L_w^*.$$

Intertwining operators and L-Toeplitz operators

Lemma

If X intertwines $\mathcal{S}$ and $\mathcal{L}_n$, then X^*X is an L-Toeplitz operator.

Proof.

We have

$$L_i X^* X L_i = X^* S_i^* S_i X = X^* X, \quad 1 \leq i \leq n.$$

□

Theorem (Popescu)

A positive L-Toeplitz operator T which is bounded below can be factored as $T = A^* A$ for some $A \in \mathcal{R}_n$.

Corollary (DLP)

If an intertwining operator $X : \mathcal{F} \to \mathcal{H}$ is bounded below, then the restriction of $\mathcal{S}$ to the range of X is unitarily equivalent to $\mathcal{L}_n$, meaning in particular that $\mathcal{S}$ has a wandering vector.

Approximate orthogonality

For $n \geq 2$, $\mathcal{L}_n$ has an infinite family of pairwise orthogonal invariant subspaces. We show that every type L free semigroup algebra has an infinite family of vectors (x_k) which are "almost orthogonal," in the sense that for fixed l, $[x_k \otimes x_l] \to 0$ and $[x_l \otimes x_k] \to 0$ as $k \to \infty$.

Idea

Suppose $\mathcal{S}^{(m)}$ has a wandering vector $\overline{x} = (x_1, ..., x_m)$. Then the cyclic subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$, generated by $S_1^{(m)}\overline{x}$ and $S_2^{(m)}\overline{x}$ respectively, are orthogonal.

If we can find a vector $\overline{u} = (u_1, ..., u_m)$ in $\mathcal{X}_1$ which is heavily concentrated in the first component, then since $[\overline{u} \otimes S_2^{(m)}\overline{x}] = 0$, the linear functional $[u_1 \otimes S_2 x_1]$ is small.

Otherwise, If we can't find such a vector, then the projection of $\mathcal{X}$ onto the last $m-1$ components is bounded below, leading to an intertwining operator which is bounded below, and hence a wandering vector for $\mathcal{S}^{(m-1)}$.

Approximate factorization

Definition

A free semigroup algebra $\mathcal{S}$ is said to have the *approximate factorization property* if, for every weak*-continuous linear functional φ on $\mathcal{S}$ and $\epsilon > 0$, there are vectors x and y such that $\|\varphi - [x \otimes y]\| < \epsilon$ and $\|x\|\|y\| \leq (1+\epsilon)\|\varphi\|$.

Observation

If X intertwines $\mathcal{S}$ and $\mathcal{L}_n$, then so does XR, for any R in $\mathcal{R}_n$.

This leads to:

Lemma

Every type L free semigroup algebra has the approximation property.

The key observation is that given operators X and Y which intertwine $\mathcal{S}$ and $\mathcal{L}_n$,

$$(S_w X\xi, Y\eta) = (L_w \xi, X^* Y\eta),$$

so $\|[X\xi \otimes Y\eta]\| = \|\xi \otimes X^* Y\eta\| \leq \|\xi\|\|X^* Y\eta\|$. In particular, if $\|X^* Y\xi - \xi\|$ is small, then the linear functional $\|[X\xi \otimes Y\eta\|$ on $\mathcal{S}$ approximates the linear functional on $\mathcal{S}$ which corresponds to the linear functional $\|[\xi \otimes \eta]\|$ on $\mathcal{L}_n$.

Conclusion

Let $\mathcal{S}$ be a type L free semigroup algebra, and let φ be a weak*-continuous linear functional on $\mathcal{S}$. With some refinement, a combination of the ideas behind the notions of "approximate orthogonality" and "approximate factorization" allows us to complete our "dual algebras" argument and construct convergent sequences (x_k) and (y_k) such that

$$\|\varphi - [x_k \otimes y_k]\| \to 0,$$

which gives $\varphi = [x \otimes y]$, where $x = \lim x_k$ and $y = \lim y_k$. This gives:

Theorem

Every type L free semigroup algebra has property $\mathbb{A}_1$.

As outlined above, this leads to:

Corollary

Every type L free semigroup algebra has a wandering vector.

Corollary

Every free semigroup algebra is reflexive, and every type L free semigroup algebra is hyper-reflexive (with distant constant 55).

Absolutely continuous representations

Definition

Let $S = [S_1, ..., S_n]$ be a row isometry. The norm-closed (non-self-adjoint) algebra generated by the S_i is called the *noncommutative disk algebra* $\mathcal{A}_n$. This was introduced by Popescu, who showed that the definition of $\mathcal{A}_n$ is independent of the choice row isometry.

Definition (DLP)

A representation σ of $\mathcal{A}_n$ on $\mathcal{B}(\mathcal{H})$ is *absolutely continuous* if every linear functional on $\mathcal{A}_n$ given by a vector state on $\sigma(\mathcal{A}_n)$ extends to a weak*-continuous linear functional on $\mathcal{L}_n$.

Theorem (Davidson, Yang)

For $n \geq 2$, if a representation σ of $\mathcal{A}_n$ is absolutely continuous, then $\sigma^{(\infty)}$ generates a type L free semigroup algebra.

Absolutely continuous representations (2)

Question

For $n \geq 2$, does an absolutely continuous representation σ generate a type L free semigroup algebra? Equivalently, if the weak* closure of $\sigma(\mathcal{A}_n)$ is isomorphic to $\mathcal{L}_n$ (and hence is non-self-adjoint), is the weak closure of $\sigma(\mathcal{A}_n)$ also non-self-adjoint?

Our methods come close, but don't quite answer this question. We suspect the answer is yes.