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ITERATED FUNCTION SYSTEMS, REPRESENTATIONS, AND HILBERT SPACE

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In this paper, we are concerned with spectral-theoretic features of general iterated function systems (IFS). Such systems arise from the study of iteration limits of a finite family of maps τ_i , $i = 1, \dots, N$, in some Hausdorff space Y . There is a standard construction which generally allows us to reduce to the case of a compact invariant subset $X \subset Y$. Typically, some kind of contractivity property for the maps τ_i is assumed, but our present considerations relax this restriction. This means that there is then not a natural equilibrium measure μ available which allows us to pass the point-maps τ_i to operators on the Hilbert space $L^2(\mu)$. Instead, we show that it is possible to realize the maps τ_i quite generally in Hilbert spaces $\mathcal{H}(X)$ of square-densities on X . The elements in $\mathcal{H}(X)$ are equivalence classes of pairs (φ, μ) , where φ is a Borel function on X , μ is a positive Borel measure on X , and $\int_X |\varphi|^2 d\mu < \infty$. We say that $(\varphi, \mu) \sim (\psi, \nu)$ if there is a positive Borel measure λ such that $\mu \ll \lambda$, $\nu \ll \lambda$, and



$$\varphi \sqrt{\frac{d\mu}{d\lambda}} = \psi \sqrt{\frac{d\nu}{d\lambda}}, \quad \lambda \text{ a.e. on } X.$$

We prove that, under general conditions on the system (X, τ_i) , there are isometries

$$S_i: (\varphi, \mu) \mapsto (\varphi \circ \sigma, \mu \circ \tau_i^{-1})$$

in $\mathcal{H}(X)$ satisfying $\sum_{i=1}^N S_i S_i^* = I$ = the identity operator in $\mathcal{H}(X)$. For the construction we assume that some mapping $\sigma: X \rightarrow X$ satisfies the conditions $\sigma \circ \tau_i = \text{id}_X$, $i = 1, \dots, N$.

We further prove that this representation in the Hilbert space $\mathcal{H}(X)$ has several universal properties.

Keywords: Hilbert space; Cuntz algebra; completely positive map; creation operators; wavelet packets; pyramid algorithm; product measures; orthogonality relations; equivalence of measures; iterated function systems (IFS); scaling function; multiresolution; subdivision scheme; singular measures; absolutely continuous measures.

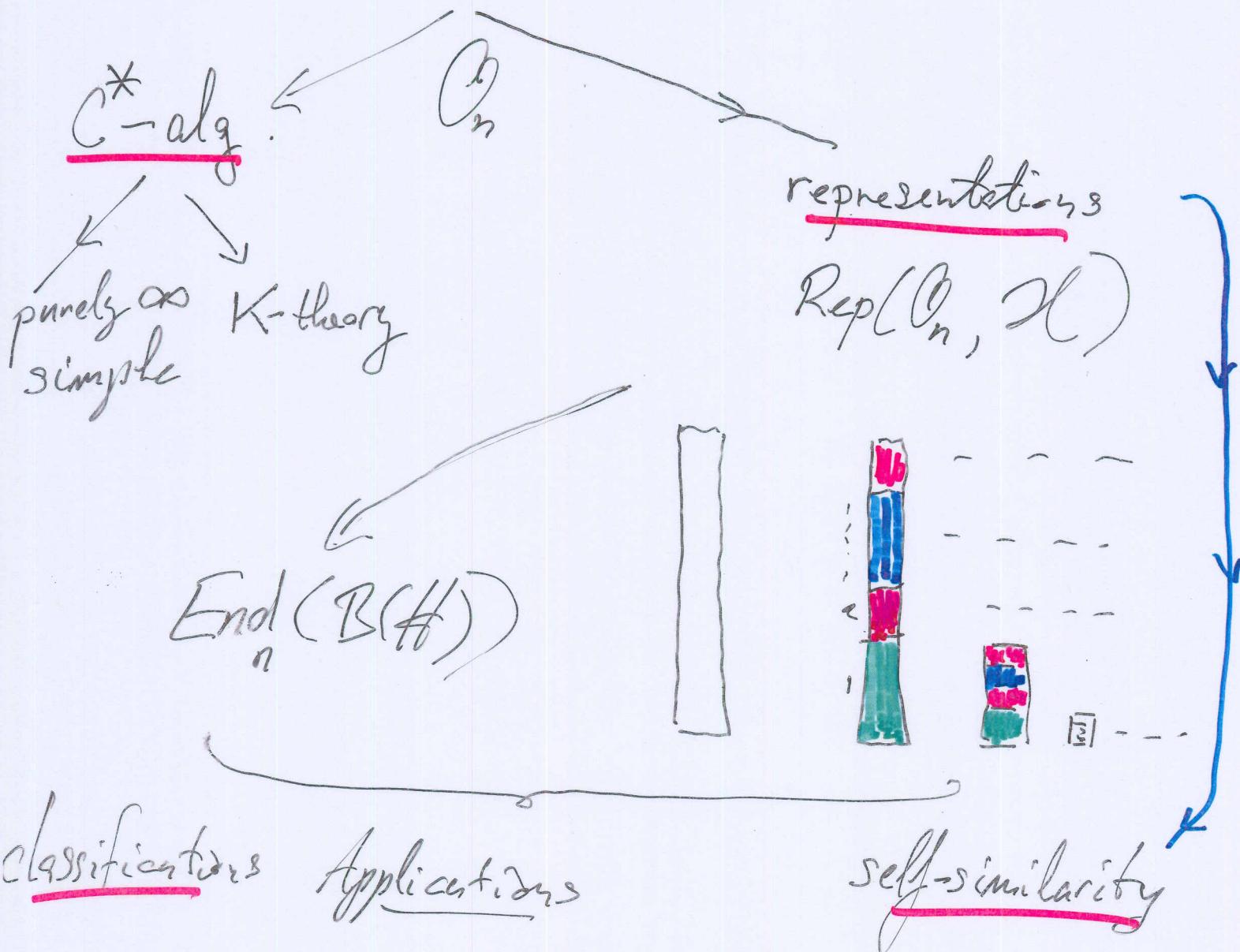
Mathematics Subject Classification 2000: 42C40; 42A16; 43A65; 42A65

1. Introduction

In this paper we are concerned with *iterated function systems* (IFS) and their representation in Hilbert space. For contractive IFS's, there is a known standard construction of a family of measures, and Hilbert spaces induced by these measures.

$$s_i^* s_j = \delta_{ij} \cdot 1, \quad \sum_i s_i s_i^* = 1$$

(1)



wavelets
constructive approximations

messiness
geometry
algorithms
measure classes

dg : Haar measure on G

$L^2(G)$ L^2 -space w.r.t. dg

$L^2(G) \ni \varphi \mapsto L_g \varphi \in L^2(G)$

REGULAR REPRESENTATION

$$(L_g \varphi)(g') := \varphi(g^{-1}g'),$$

$\forall g, g' \in G$

Glimm

$2\mathcal{C}$ C^* : Does $\text{Irr } 2\mathcal{C}$

? have a Borel cross section?

• Good
• Bad

\mathcal{L} : Complex Hilbert sp.

Functor

$\mathcal{L} \rightarrow C^*$ -alg. $2\mathcal{C}(\mathcal{L})$

$l \in \mathcal{L}$

$v_l \in 2\mathcal{C}$

Example

Funz

$$\|\ell\|_{\mathcal{L}} = \|\nu_\ell\|_{\mathcal{L}^*} \quad \forall \ell \in \mathcal{L}$$

Hilbert

C^*

$$2\mathcal{C}(\mathcal{L}) := C^*-completion (\nu_\ell)_{\ell \in \mathcal{L}}$$

$$\nu_{\ell_1}^* \nu_{\ell_2} = \langle \ell_1, \ell_2 \rangle_{\mathcal{L}} \mathbf{1} \quad (1)$$

$$(e_i) \text{ OND in } \mathcal{L} : \sum_i v_i v_i^* = \mathbf{1} \quad (2)$$

Cuntz (1977, Thm) : $\bullet 2\mathcal{C}(\mathcal{L})$ is simple,

- He found its K-theory,
- not Glimm type.

• \exists special family of pure states:

$$\pi(\nu_\ell) = \nu_\ell, \quad \ell \in \mathcal{L}, \quad w \in \mathcal{L}, \quad \|w\| = 1$$

$$\nu_\ell^* \Omega_w = \langle \ell | w \rangle_{\mathcal{L}} \Omega_w, \quad \forall \ell \in \mathcal{L}.$$

$(s_i) \in \text{Rep}(\mathbb{Q}_n, \mathcal{H})$



$$\sigma(A) := \sum_i s_i A s_i^*$$

$\sigma \in \text{End}_{\mathcal{H}}(\mathcal{B}(H))$

$$\sigma(AB) = \sigma(A)\sigma(B)$$

$$\sigma(I) = I$$

$$\sigma(A^*) = \sigma(A)^*$$

Completely (Arveson)

$$\sigma(A)s_i s_j^* = s_i s_j^* \sigma(A)$$

σ

$\sigma(\mathcal{B}(H))$

$\mathcal{B}(H) \cap \sigma(\mathcal{B}(H))'$

e_{ij}

*relative
complement*

I_n



S_i

st

$$e_{ij} = s_i s_j^*$$

(2)

IFS

$\tau_i : X \rightarrow X$ complete metric

contractive: $\exists c < 1 : d(\tau_i x, \tau_i y) \leq c d(x, y)$

probabilities: $p_i > 0 \quad \forall i, \forall x, y$

Theorem (Hutchinson '81) \exists ! probability Borel μ
sc.

$$\mu = \sum_i p_i \mu \circ \tau_i^{-1}$$

$$(\mu \circ \tau^{-1})(E) = \mu(\tau^{-1}(E))$$

$$\tau^{-1}(E) = \{x \in X \mid \tau(x) \in E\}$$

Examples

Affine

$A \quad n \times n \quad$ over \mathbb{Z} , $\det A \neq 0$,

$\mathbb{X} := \mathbb{R}^n / \mathbb{Z}^n \hookrightarrow x \mapsto Ax, x \in \mathbb{R}^n$

$\sigma: \quad \text{mod } \mathbb{Z}^n$

IFS fractal

→ symbol space

$$(\tau_i^F)_{i \in A}$$

$$\tau_i^F : F \rightarrow F$$

$$X = \prod A = \{(i_1 i_2 \dots) \mid i \in A\}$$

$$(i_1 i_2 \dots) \xrightarrow{\sigma} (i_2 i_3 \dots)$$

$$(i_1 i_2 \dots) \xrightarrow{\tau_i^F} (i_1 i_2 \dots)$$

$$\sigma \circ \tau_i^F = \text{id}_X$$

$$\pi : X \rightarrow F$$

$$\pi(i_1 i_2 i_3 \dots) := \bigcap_n \underbrace{\tau_{i_1}^F \tau_{i_2}^F \dots \tau_{i_n}^F}_{\text{truncation at } n}(F)$$

$$\pi \circ \tau_i^F = \tau_i^F \circ \pi$$

symbol

~~Ex~~

$$X := \prod_{i \in \mathbb{N}} \{\pm 1\}, \quad s \in \mathbb{R}^+$$

$\tau_{\pm}(x) = s(x \pm 1)$

Bernoulli
 measure
 coin toss

$\omega \in X$

$$\pi(\omega) = S(\omega) := \sum_{i \in \mathbb{N}} \omega_i s^i = \sum_L (\pm) s^i$$

random variable

$$\mu_s((-\infty, x]) = P(\{\omega \mid S(\omega) \leq x\})$$

$$s = \frac{1}{2}$$

Half

$$s = \frac{1}{3}$$

middle third Cantor

$$s = \frac{1}{4}$$

$$\frac{1}{4}$$

on B

\exists n distinct branches of the inverse (3)

$$\# \left(\frac{\mathbb{Z}^n}{A\mathbb{Z}^n} \right) = N := |\det A|$$

$$(b_i)_{i=1}^N$$

$$\tau_i(x) := \tilde{A}(x + b_i)$$



$$\sigma \circ \tau_i = \text{id}_{\mathbb{X}}$$

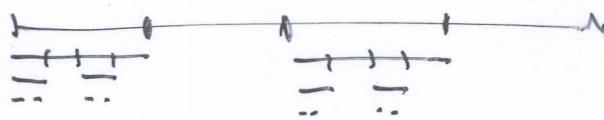
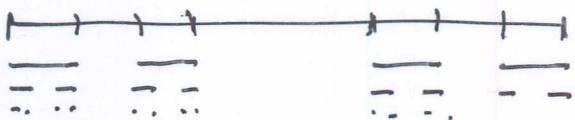
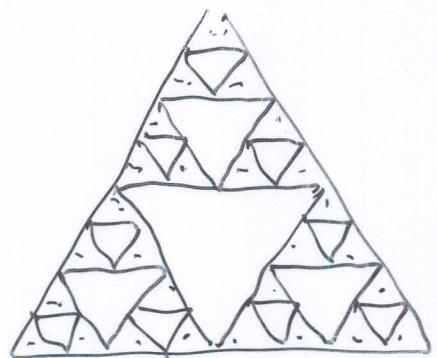
μ = Haar-measure on the torus

Select a subset $\hat{C} \subset \{b_i\}$

μ_C fractal measure

Ex

$$p_i = \sqrt{\frac{1}{\# C}}$$



$$\frac{x}{3}$$

$$\frac{x+2}{3}$$

CANTOR

$$\frac{x}{4}$$

$$\frac{x+2}{4}$$

MEASURES



S_i Example $n=2$

Rep($\mathcal{O}_2, \mathcal{H}$)⁽⁴⁾

$$\begin{cases} f \rightarrow f(z^2) \\ f \rightarrow z f(z^2) \end{cases}$$

\mathcal{H} = the Hardy space H^2

$$\begin{cases} f \rightarrow f(z^3) \\ f \rightarrow z f(z^3) \end{cases}$$

Subspace of H^2

$$\begin{cases} f \rightarrow f(z^4) \\ f \rightarrow z f(z^4) \end{cases}$$

Subspace of H^2
S1

$L^2(\mu_4)$

Fourier bases for fractal measures

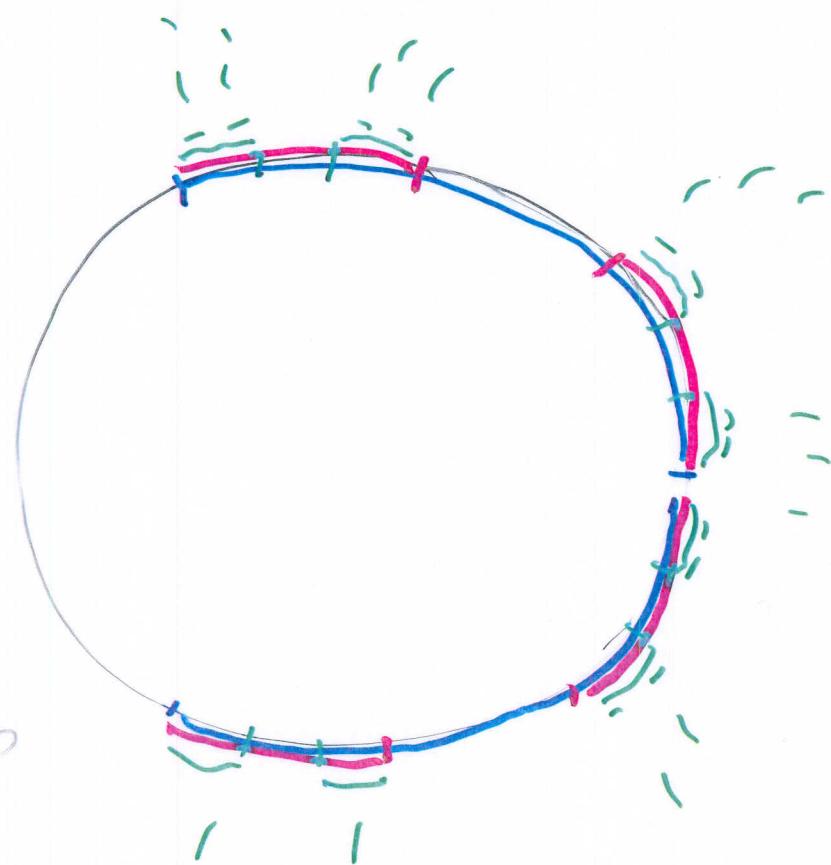
Ex $L^2(\mu_4)$ $e_j(x) = e^{i\theta\pi j x}$

ONB in $L^2(\mu_4)$ $\Lambda = \{0, 1, 4, 5, 16, 17, 80, \dots\}$

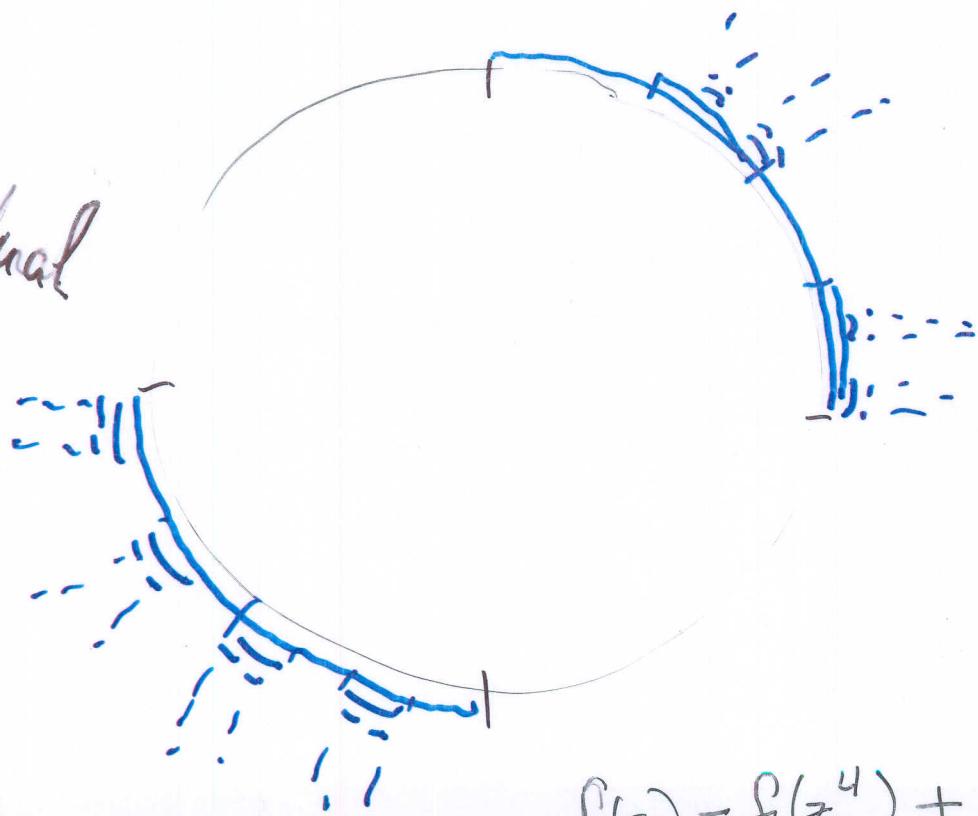
$$\left\{ \sum_{i=0}^{\text{finite}} a_i 4^i \right\}, \quad a_i \in \{0, 1\}.$$

μ_3 C_3

Fourier

No!
dual? μ_4 Fourier
dual

YES!

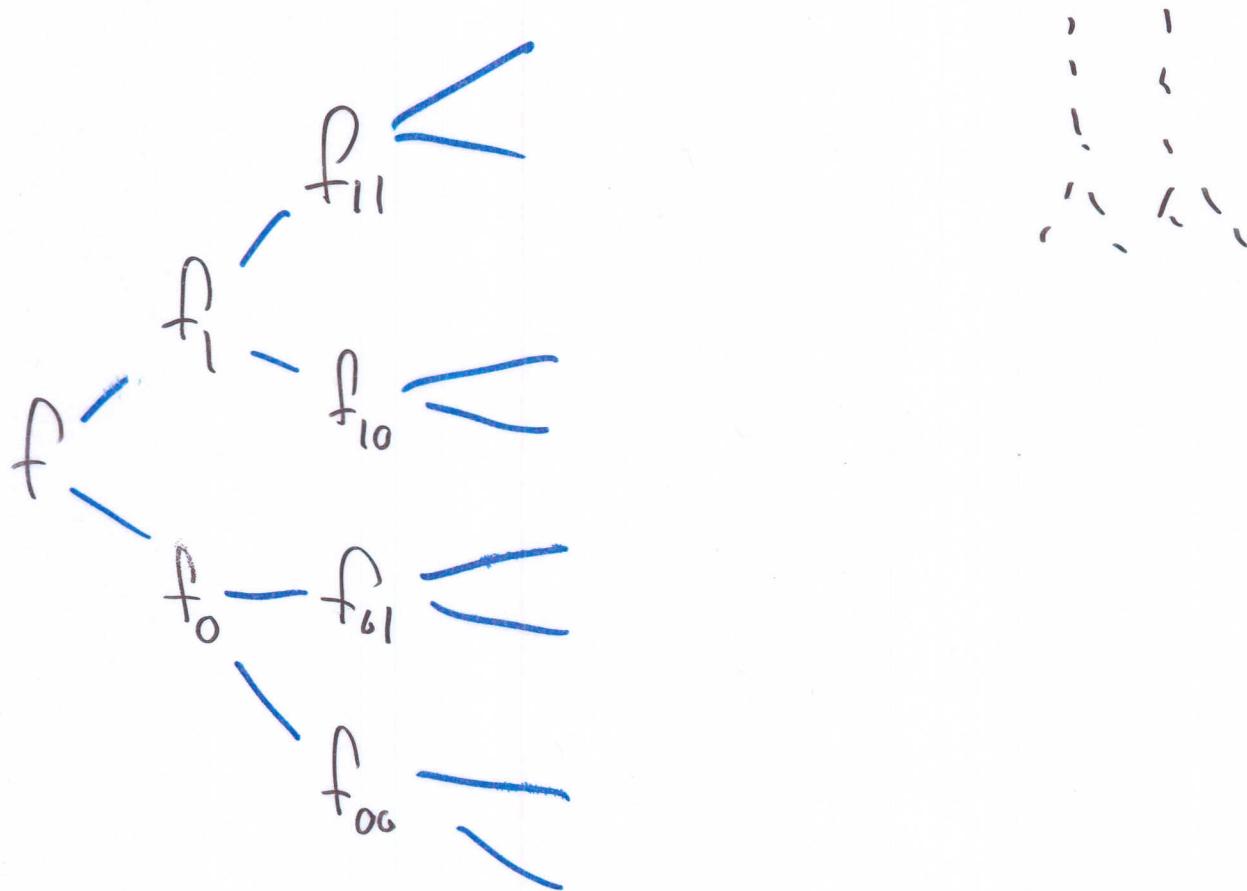
 C_4 

$$f(z) = f_0(z^4) + z f_1(z^4)$$

$H_4 \subset H^2 = \text{the Hardy space}$

$\forall f \in H_4 \exists! f_0, f_1 \in H_4$ st

$$f(z) = f_0(z^4) + z f_1(z^4)$$



$$f_{00}(z^{16}) + z^4 f_{01}(z^{16}) + z f_{10}(z^{16}) + z^5 f_{11}(z^{16})$$

The

$$H_4 \underset{\text{Fourier}}{\simeq} L^2(\mu_4)$$

$$f(x) = \sum_{\lambda \in \Lambda} a(\lambda) e^{i 2\pi \lambda x}$$

$$e_\lambda(x) = e^{i 2\pi \lambda x} = z^\lambda$$

$$f(z) = \sum_{\lambda} a(\lambda) z^\lambda \in H_4^{\perp} \subset H^2$$

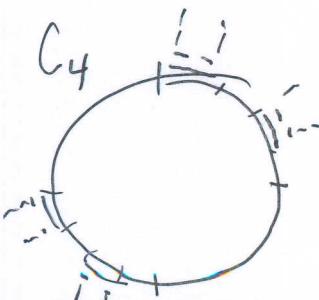
FATCG theory $\exists \tilde{f} \in L^2(T, L^2)$

a.e.

non-tay

$$\sum_{\lambda \in \Lambda} |a(\lambda)|^2 = \int_0^1 |\tilde{f}(x)|^2 dx$$

$$= \int |\tilde{f}(x)|^2 d\mu_4(x)$$



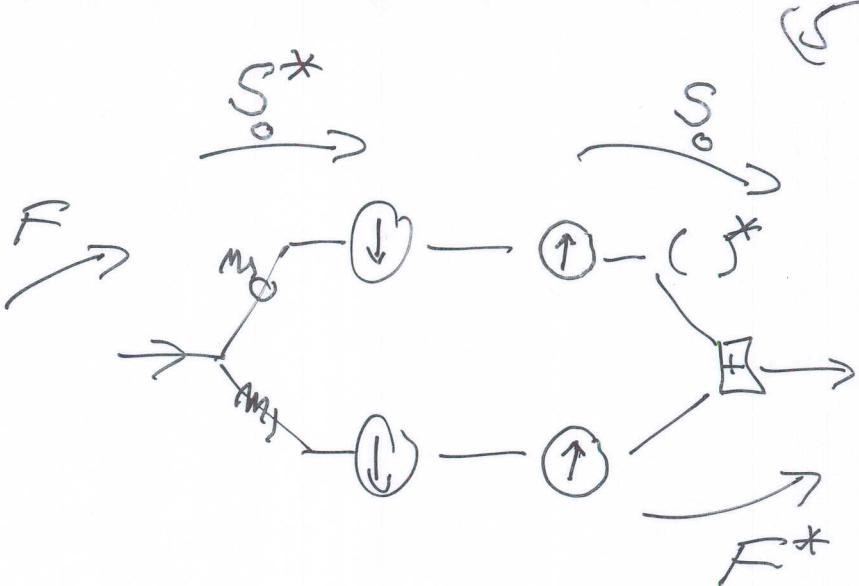
Cantor set

$\rightarrow C_4$

Wavelets

$$\sum_i S_i S_i^* = I$$

bands \perp

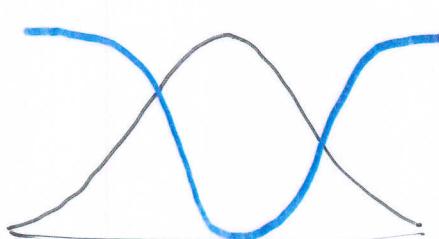


(x_n) signal $n \in \mathbb{Z}$ time

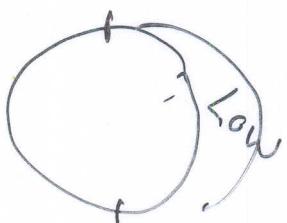
$$(\mathbb{Z})^* = \mathbb{T}$$

$$\sum x_n z^n \quad z = e^{i 2\pi \theta}$$

$$|m_i|^2$$



$$\frac{\text{Law}}{\theta=0}$$



$$f \mapsto m_i(z) f(z^2) \quad \mathcal{O}_2$$

$${}^0 \leftarrow f \rightarrow m_i(z) f(z^A)$$



$$\overline{\mathbb{T}}^L$$

matrix operat

Complex form

d-torus $\overline{\mathbb{T}}^d$

$$\left\{ (z_1, \dots, z_d) \in \mathbb{C}^d \mid |z_k| = 1, \quad k=1, \dots, d \right\}$$

$$\overline{\mathbb{T}}^d = \underbrace{\overline{\mathbb{T}} \times \overline{\mathbb{T}} \times \cdots \times \overline{\mathbb{T}}}_d$$

d copies of the circle group

Simple fact

$$\overline{\mathbb{T}}^d \cong \mathbb{R}^d / \underline{\mathbb{Z}}^d$$

$\underline{\mathbb{Z}}^d$: the d-integer lattice
in \mathbb{R}^d .

$$I = (i_1 i_2 \dots i_k) \quad \bar{J} = (j_1 j_2 \dots j_k)$$

finite words

$$S_I := s_{i_1} s_{i_2} \dots s_{i_k}$$

$$S_I^* := s_{i_k}^* \dots s_{i_1}^*$$

Subcategories of \mathcal{Q}_N

- $S_I S_J^*$ words of same length
UHF N^∞

$$M_N \otimes M_N \otimes \dots$$

$$e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes \dots \otimes e_{i_k j_k}$$

- $S_I S_I^*$ MASA

completely finite & projected

(2)

The wavelet representation, $N=2$

$$(S_i f)(z) = m_i(z) f(z^2), \quad i=1, 2$$

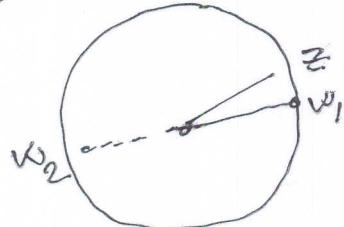
$$f \in L^2(\mathbb{R})$$



$$(S_i^* f)(z) = \frac{1}{2} \sum_{w \in \{\pm\sqrt{z}\}} \overline{m_i(w)} f(w)$$

$w^2 = z$

$w \in \{\pm\sqrt{z}\}$



Warning: The operators S_1, S_1^* are not diagonalized in this representation!

$$(S_i S_2^* f)(z) = \frac{1}{2} m_i(z) \left[\overline{m_2(z)} f(z) + \overline{m_2(-z)} f(-z) \right]$$

PERMUTATIVE REPS

S_i^* permutes the vectors in some basis

$$\rho \in \text{Rep}(\mathcal{O}_N, \mathbb{H})$$

$$\rho = \sum \oplus \text{ irreducible repn's of } \mathcal{O}_N$$

$$\sigma \in \text{IrrRep}(\mathcal{O}_N, \mathbb{H})$$

$$\sigma|_{\text{UHF}} = \sum \oplus \text{ irreducible repn's of UHF}$$



Permutative

$\exists \alpha \in \mathbb{N} \quad (\epsilon_\alpha)_{\alpha \in \mathbb{N}}$

in \mathcal{H} s.t. for

$\forall i \forall j \alpha \in \mathbb{N} \quad S_i e_\alpha \in \{e_\beta \mid \beta \in \mathbb{N}\}$

$$S_i e_\alpha = e_{\sigma_i(\alpha)}$$

$$\sigma_i : \mathbb{N} \rightarrow \mathbb{N} \quad 1-1$$

$$\sigma_i(\mathbb{N}) \cap \sigma_j(\mathbb{N}) = \emptyset \quad i \neq j$$

$$\bigcup_i \sigma_i(\mathbb{N}) = \mathbb{N}, \quad R(\sigma_i(\mathbb{N})) = V$$

~~Def~~ $\mathbb{N} \xrightarrow{\sigma} \{1, 2, \dots, n\}^\infty$

$$\sigma : \alpha \longrightarrow (i_1 i_2 i_3 \dots)$$

$$S_{i_k}^* S_{i_{k+1}}^* \dots S_{i_l}^* e_\alpha \neq 0,$$

Example $\mathbb{N} = \mathbb{Z}^n$

(5)

• Equivalence relation on the index set P for a permutation map σ ,

• $\alpha \sim \beta$: $\exists k, n_0$ st

$$\sigma(\alpha)_{i+k} = \sigma(\beta)_{i+k}, \forall i > n_0$$

• $\alpha \approx \beta$ $\exists n_0$ st

$$\sigma(\alpha)_i = \sigma(\beta)_i, \forall i > n_0$$

Lemma

• $\alpha \sim \beta \Leftrightarrow \exists k, l$ st. $R_\alpha^k = R_\beta^l$

• $\alpha \approx \beta \Leftrightarrow \exists k$ st. $R_\alpha^k = R_\beta^k$

A expansion $n \times n$, $N := |\det A|$

\mathcal{O}_N $\frac{\mathbb{R}^n}{\mathbb{Z}^n} \cong \mathbb{T}^n$ in complex coords

$$\lambda \in \underline{\mathbb{Z}}^n \quad z^\lambda := z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_n^{\lambda_n}$$

z^A is $\mathbb{R}^n \ni x \mapsto Ax$
passed to $\frac{\mathbb{R}^n}{\mathbb{Z}^n} = \mathbb{T}^n$

Pick $d_1, d_2, \dots, d_N \in \underline{\mathbb{Z}}^n \setminus A\underline{\mathbb{Z}}^n$

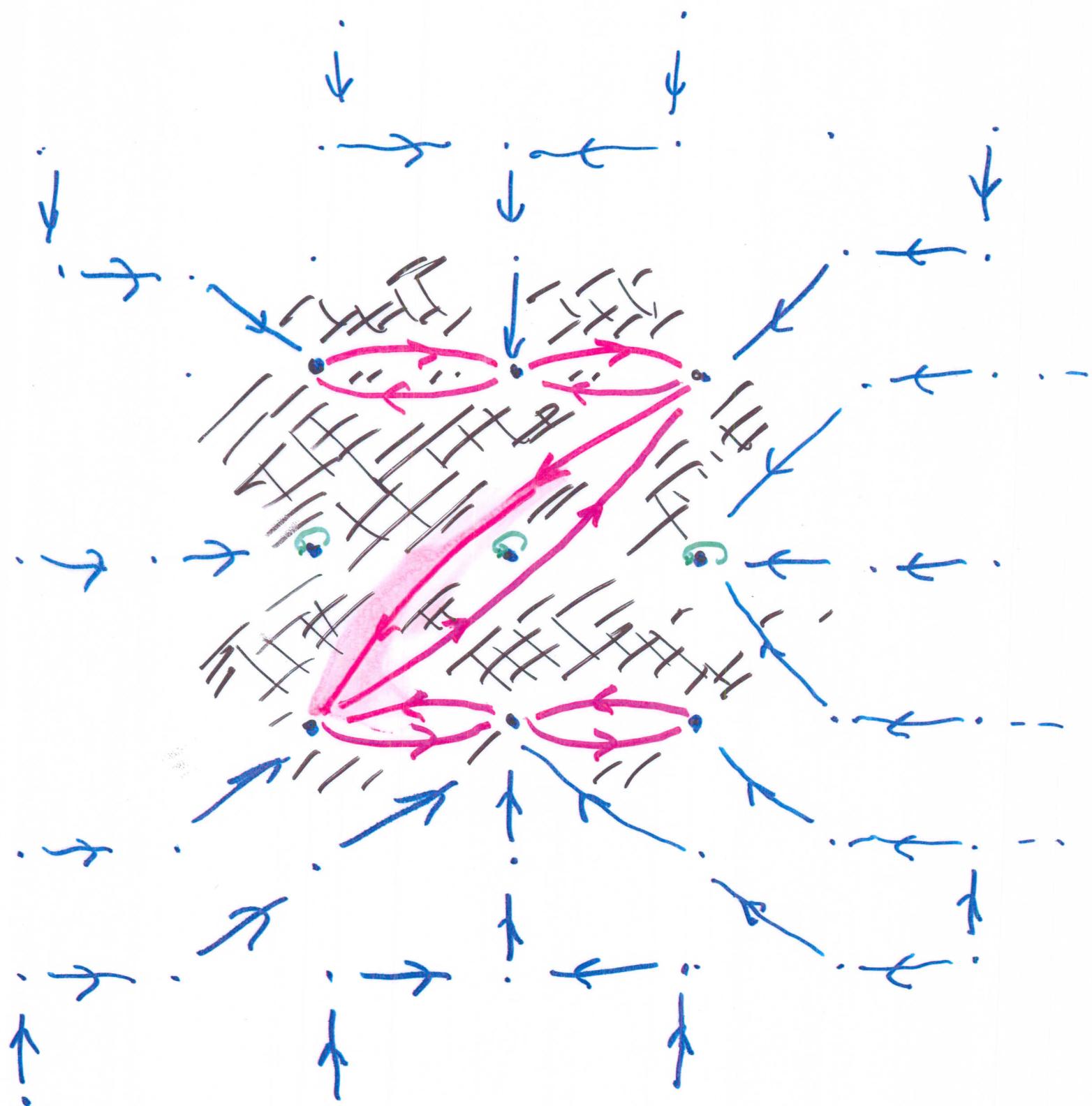
Set

$$(S_i f)(z) = z^{d_i} f(z^A)$$

$$\bigcap_k A^k \underline{\mathbb{Z}}^n = \{0\}$$

$\exists R: \underline{\mathbb{Z}}^n \rightarrow \underline{\mathbb{Z}}^n$ st

$$R(\underbrace{d_i + A\lambda}_{\in \underline{\mathbb{Z}}^n}) = \lambda \in \underline{\mathbb{Z}}^n$$



multi-wavelets

$$N = |\det A|$$

$A \in \mathbb{R}^{d \times d}$ over \mathbb{Z}

$$|\lambda| > 1$$

$$\psi \in L^2(\mathbb{R}^d)$$

$$N^{j/2} \chi_i(A^j x - k)$$

$$j \in \mathbb{Z}, k \in \mathbb{Z}^d \rightarrow x, \xi \in \mathbb{R}^d$$

$$N^{j/2} \chi(A^j \xi) e^{i \pi k \cdot A^j \xi}$$

? $\hat{\psi} = \chi_E, E \subset \mathbb{R}^d$

is a wavelet (Cheney)
↑
 E tiles \mathbb{R}^d by A^j et al
and E tiles \mathbb{R}^d by \mathbb{Z}^d +

Julia sets

$$z \mapsto \underbrace{z^2 + c}_{\delta}, \quad \tilde{\tau}_\pm : z \mapsto \pm \sqrt{z - c}$$

$$\sigma \circ \tau_\pm = \text{id} \quad \exists! \mu$$

st. $\mu = \frac{1}{2} \sum_{\pm} \mu \circ \tilde{\tau}_\pm^{-1}$

or

$$z \rightarrow R(z)$$

↑ rational

branches of inverse

$$\text{Supp}(\mu) = \text{Julia set}$$

$$:= \left(\bigcup \{ O \mid R^k|_O \text{ is normal} \} \right)^c$$

$\mathcal{H}(\Sigma)$

$$(\varphi, \mu) \sim$$

$$(\varphi, \mu) \underset{D}{\sim} (\chi, \nu)$$

$$\exists \lambda \text{ s.t. } \mu \ll \lambda, \nu \ll \lambda$$

and

$$\varphi \sqrt{\frac{d\mu}{d\lambda}} = \chi \sqrt{\frac{d\nu}{d\lambda}} \quad \text{a.e. } \lambda$$

$$\text{Def: } \text{class}(\varphi, \mu) =: \varphi \sqrt{d\mu}$$

$$\langle \varphi \sqrt{d\mu}, \chi \sqrt{d\nu} \rangle := \int \bar{\varphi} \chi \sqrt{\frac{d\mu}{d\lambda}} \sqrt{\frac{d\nu}{d\lambda}} d\lambda$$

sum +

$$\left(\varphi \sqrt{\frac{d\mu}{d\lambda}} + \chi \sqrt{\frac{d\nu}{d\lambda}}, \lambda \right)$$

Fix μ

$$\{(\varphi, \mu)\} \subset L^2(\mu)$$

$$\left\| \varphi \sqrt{d\mu} \right\|_{\mathcal{H}(X)}^2 = \int |\varphi|^2 d\mu = \|\varphi\|_{L^2(\mu)}^2$$

$\mu \neq \nu$ mutually singular

$$H(\mu) \perp H(\nu)$$

as subspaces in $\mathcal{H}(X)$

$H(\mu)$ arise as cyclic subspaces

Ex $A = A^\dagger$ on \mathcal{K} s.t. $k_i \in \mathbb{R}$

$\Rightarrow \exists \mu_1$ st

$$\|\varphi(A)k_1\|_{\mathcal{K}}^2 = \int |\varphi|^2 d\mu_1$$

THEOREM

(9)

Given

$$\rho \in \text{Rep}(\mathbb{Q}_N, \mathcal{K})$$

\exists some system X, σ_i, τ_i

$$\sigma: X \rightarrow X, \quad \tau_i: X \rightarrow X, \quad i = 1, \dots, N$$

$$\sigma \circ \tau_i = \text{id}_{\underline{X}}$$

$$\exists \pi^{\text{Univ}} \in \text{Rep}(\mathbb{Q}_N, \mathcal{H}(\underline{X}))$$

$$\exists W: \mathcal{K} \rightarrow \mathcal{H}(\underline{X})$$

s.t.

$$W\rho(s_i) = \pi^{\text{Univ}}(s_i) W$$

 f_i

$$\pi^{\text{Univ}}(s_i) (\varphi, \mu) = (\varphi \circ \sigma, \mu \circ \tau_i^{-1})$$

acting as elements in $\mathcal{H}(X)$

$$\rho \sqrt{d\mu} = \text{class}(\varphi, \mu) \in \mathcal{H}(X)$$

then $\int \rho^{\mu \circ \tau_i} << \mu$

Set

$$S_i^\mu \rho := \varphi \circ \sigma \sqrt{\frac{d\mu \circ \tau_i^{-1}}{d\mu}}$$

$$W^{(\mu)} \varphi := (\rho, \mu)$$

then

$$W^{(\mu)} : \mathcal{E}(\mu) \rightarrow \mathcal{H}(X)$$

is isometric

(10)

and

$$E(\mu) \xrightarrow{W^{(p)}} \mathcal{H}(X)$$
$$S_i^{(p)} \downarrow \quad \downarrow \pi^{\text{Univ}}_{\sigma_i}$$

$$E(\mu) \xrightarrow{W^{(p)}} \mathcal{H}(X)$$

$$(S_i^{(p)}) \in \text{Rep}(\mathbb{Q}_\ell, E(\mu))$$

Sit $P_\mu := \frac{d\mu \circ \bar{\tau}_i^{-1}}{d\mu}$

then

$$S_i^{(p)} = \varphi \circ \bar{\tau}_i \left(P_\mu \circ \bar{\tau}_i \right)^{-\frac{1}{2}}$$

$$\forall \varphi \in E(\mu) \quad H_i$$