

Commutant lifting for  
commuting row contractions

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joint work with Tyea He

# Contraction

Thm (Sz Nagy-Foias)  $T \in \mathcal{B}(H)$ ,  $\|T\| \leq 1$   
 $\exists$  unique minimal unitary  $d|T|^n$

$$U \cong \begin{pmatrix} x & 0 & 0 \\ x & T & 0 \\ x & * & x \end{pmatrix}$$

Model  $U = \text{bilateral shift on } \ell^2(\mathbb{Z})$

Cor (von Neumann inequality)  
 $\|T\| \leq 1$ ,  $P(z)$  polynomial

$$\|P(T)\| \leq \|P(U)\| = \sup_{|z| \leq 1} |P(z)|$$

Thm (Sz. Nagy - Foaia's CLT)

$$\|T\| \leq 1, \text{ unitary dil}^n \quad U = \begin{pmatrix} * & 0 & 0 \\ * & T & 0 \\ * & * & * \end{pmatrix}$$

$$\|X\| = 1 \quad TX = XT$$

$$\text{Then } \exists Y = \begin{pmatrix} * & 0 & 0 \\ * & X & 0 \\ * & * & * \end{pmatrix}, \quad UY = YU, \quad \|Y\| = 1.$$

Cor (Ando)

$$\|T\| \leq 1, \quad \|X\| \leq 1, \quad TX = XT$$

$$\text{Then } \exists \text{ unitaries } U \cong \begin{pmatrix} * & 0 & 0 \\ * & T & 0 \\ * & * & * \end{pmatrix} \quad V \cong \begin{pmatrix} * & 0 & 0 \\ * & X & 0 \\ * & * & * \end{pmatrix}, \quad UV = VU.$$

- There is no CLT for 2 (or more) commuting contractions.

## (non-comm) Row Contractions

Thm (Frazho - Bunce - Popescu)

$$T = [T_1 \ T_2 \ \dots \ T_n] \quad TT^* = \sum T_i T_i^* \leq I \quad (\text{i.e. } \|T\| \leq 1)$$

Then  $\exists$  unique minimal isometric dilation

$$S = [S_1 \ S_2 \ \dots \ S_n] \quad S_i \cong \begin{pmatrix} T_i & 0 \\ * & * \end{pmatrix} \quad S_i^* S_j = \delta_{ij} I$$

Model  $\mathbb{F}_n^+$  free semigroup (words in  $1, \dots, n$ )

Fock space  $\ell^2(\mathbb{F}_n^+)$ ,  $L_i \xi_w = \xi_{iw}$   $i, w \in \mathbb{F}_n^+$

$L = [L_1 \ L_2 \ \dots \ L_n]$  is a row isometry

Cor (Popescu's vN.  $\leq$ )  $T = [T_1 \ \dots \ T_n] \quad \|T\| \leq 1$

$p$  non-comm. pol<sup>e</sup>. Then

$$\|p(T_1, \dots, T_n)\| \leq \|p(L_1, \dots, L_n)\|$$

Thm (Popescu's CLT)

$$T = [T_1 \dots T_n] \quad \|T\| \leq 1, \quad T_i X = X T_i, \quad 1 \leq i \leq n, \quad \|X\| = 1$$

$$S = [S_1 \dots S_n] \text{ row isometric dil}^e \text{ of } T \quad \|Y\| = 1$$

$$\text{Then } \exists Y = \begin{pmatrix} X & 0 \\ * & * \end{pmatrix}, \quad S_i Y = Y S_i, \quad 1 \leq i \leq n$$

Popescu's proof is to point out that the Sz. Nagy - Foias' proof works verbatim.

## Commuting Row Contractions

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Model  $H_n^2 =$  Symmetric Fock space = Drury-Arveson Space

RKHS over  $\mathbb{B}_n$   $\{k_\lambda\}_{\lambda \in \mathbb{B}_n}$

orthogonal basis  $\mathcal{J}^k$ ,  $k \in \mathcal{N}_0^n$  (not normalized!)

$f \in H_n^2$   $f = \sum a_k \mathcal{J}^k$ ,  $f(\lambda) = \sum a_k \lambda^k = \langle f, k_\lambda \rangle$

multipliers  $M_\eta f = hf$

$M = [M_{z_1}, M_{z_2}, \dots, M_{z_n}]$

Thm (Drury's VN $\leq$ )

$T = [T_1, \dots, T_n]$ ,  $\|T\| \leq 1$ ,  $T_i T_j = T_j T_i$ ,  $1 \leq i, j \leq n$

$P \in \mathcal{C}[\mathbb{Z}_1, \dots, \mathbb{Z}_n]$

Then  $\|P(T)\| \leq \|P(M)\|$ .

Thm (Müller-Vasilescu, Arveson)

$$T = [T_1, \dots, T_n], \|T\| \leq 1, T_i T_j = T_j T_i, 1 \leq i, j \leq n$$

Then  $\exists$  unique minimal dilation to commuting  $n$ -tuple

$$\tilde{T} = \begin{pmatrix} T & 0 \\ * & * \end{pmatrix} \text{ s.t. } \tilde{T}_i \cong M_{2_i}^{(2)} \oplus U_i$$

where  $U_i$  normal,  $\sum_{i=1}^n U_i U_i^* = I$  ) Spherical  
unitary

Thm (Arveson) The operator alg. of the universal

commuting row contraction has  $C^*$ -envelope

$$C^*(M) = C^*(M_{2_1}, \dots, M_{2_n}) \text{ and}$$

$$0 \rightarrow \mathcal{K} \rightarrow C^*(M) \rightarrow C(\partial B_n) \rightarrow 0$$

Thm (D-, Le) CLT for comm. row cont.

$$T = [T_1 \dots T_n], \|T\| \leq 1, T_i T_j = T_j T_i, 1 \leq i, j \leq n.$$

Arveson dilation  $\tilde{T}_i = \begin{pmatrix} T_i & 0 \\ * & * \end{pmatrix} \cong M_{2, i}^{(n)} \oplus U_i$

$$X \in B(K), \|X\| = 1, T_i X = X T_i \quad 1 \leq i \leq n$$

$$\text{Then } \exists Y = \begin{pmatrix} X & 0 \\ * & * \end{pmatrix}, \|Y\| = 1, \tilde{T}_i Y = Y \tilde{T}_i, 1 \leq i \leq n.$$

# Elements of the proof of CLT

$$T = [T_1 \dots T_n] \leftrightarrow X$$

Dilate to  $\mathbb{T} \cong M^{(\infty)} \oplus U$  on  $\mathcal{H}$

Dilate to  $\mathbb{T} \cong L^{(\infty)} \oplus S$   $S$  Contra type  $\leftrightarrow \mathbb{Z}$  by Popescu CU

Try  $Y = P_{\mathcal{K}} Z|_{\mathcal{K}}$

$A \in \mathcal{R}_n = \overline{\text{alg}\{R_1, \dots, R_n\}}$  WOT  
 Popescu, D-P.H.s

Facts 1)  $A \in L_1, \dots, L_n \Leftrightarrow A \in \mathcal{R}_n = \overline{\text{alg}\{R_1, \dots, R_n\}}$   
 o.  $f_n^{(x)}$   $\cong M_n(\mathbb{R}_n)$

2)  $K_n, \mathcal{R}_n$  leave  $H_n^2$  coinvariant  
 compression of  $f_n$  to  $H_n^2$  is a complete quotient  
 map onto  $\text{Mult}(H_n^2)$   
**D-P.H.s**

3)  $U$  spherical isom,  $A = \sum U_i A U_i^* \Rightarrow A \in C^*(\{U_1, \dots, U_n\})'$   
**Davie-Jewell** (we give simpler proof)

## The Ball-Trent-Vinnikov CLT

$H$  irreducible complete NP kernel

Characterized by Quiggen and McCullough

Thm (Agler-McCarthy) Can identify  $X$  with  
a subset of  $\mathbb{B}_n$  ( $n$  large enough) s.t.

$$H \cong \text{span} \{k_\lambda : \lambda \in X\} \subseteq H_n^2$$

Thm (BTV)  $M \subset H^{(\alpha)}$  co-invariant for  $M_c H(H)^{(\alpha)}$

$$X \in \mathcal{B}(M), \|X\| = 1, X P_M M_n^{(\alpha)} = P_M M_n^{(\alpha)} X \quad \forall n$$

$$\text{Then } \exists Y \in \mathcal{B}(H^{(\alpha)}), \|Y\| = 1, P_M Y = X P_M$$

$$\text{s.t. } Y M_n^{(\alpha)} = M_n^{(\alpha)} Y$$

BTV is a special case of our result

since  $\exists \tilde{Y}$  dilating  $X$ ,  $\tilde{Y} \leftrightarrow \text{Mult}(H_n^2)^{(a)}$

$$\therefore \tilde{Y} \in M_{\mathcal{Q}}(\text{Mult}(H_n^2))$$

$$\Rightarrow Y = P_{H^{(a)}} \tilde{Y} |_{H^{(a)}} \in M_{\mathcal{Q}}(\text{Mult}(H)) = (\text{Mult}(H))^{(a)}$$

Proof of "pure case"

$$T_c = P_M M_{z_i} | M$$

dilates to  $M^{(a)} = [M_{z_1}^{(a)} \dots M_{z_n}^{(a)}]$

dilates to  $L^{(a)} = [L_1^{(a)} \dots L_n^{(a)}]$

$$\leftrightarrow X \quad \|X\|=1$$

$$\leftrightarrow Z \quad \|Z\|=1$$

Popescu CLT

$$\bullet (k_n^{(a)})' = M_{\mathcal{Q}}(R_n)$$

$$\bullet \delta_n, R_n \text{ leave } H_n^2 \text{ coinvar. } \circ Y = P_{H_n^2} Z |_{H_n^2} \text{ works}$$