

Canonical functional-model realizations of contractive multipliers of the Drury-Arveson space

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The classical case

Notation:

\mathcal{U}, \mathcal{Y} – Hilbert spaces,

$\mathcal{L}(\mathcal{U}, \mathcal{Y})$ – the algebra of bounded operators from \mathcal{U} into \mathcal{Y} ,

\mathbb{D} – the unit disk,

$H^2_{\mathcal{U}}$ – the Hardy space of \mathcal{U} -valued functions of \mathbb{D} ,

$\mathcal{S}(\mathcal{U}, \mathcal{Y})$ – the Schur class of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued functions analytic and contractive valued on \mathbb{D} .

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- (5) The kernel $\hat{K}_S(z, \zeta) = \begin{bmatrix} K_S(z, \zeta) & \frac{S(z) - S(\zeta)}{z - \zeta} \\ \frac{S(z)^* - S(\zeta)^*}{\bar{z} - \bar{\zeta}} & \tilde{K}_S(z, \zeta) \end{bmatrix}$ is positive on \mathbb{D} .

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(6) There exists a Hilbert space \mathcal{H} and a unitary (isometric, coisometric, contractive) operator $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix}$

such that $S(z) = D + zC(I - zA)^{-1}B$ ($z \in \mathbb{D}$).

de Branges-Rovnyak coisometric realization

Let $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and let $\mathcal{H}(K_S)$ be the RKHS with RK $K_S(z, \zeta)$.

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$$\begin{aligned} A: f(z) &\mapsto \frac{f(z) - f(0)}{z}, & B: u &\mapsto \frac{S(z) - S(0)}{z} u, \\ C: f(z) &\mapsto f(0), & D: u &\mapsto S(0)u. \end{aligned}$$

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- (3) If $\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} : (\mathcal{X} \oplus \mathcal{U}) \rightarrow (\mathcal{X} \oplus \mathcal{Y})$ is another colligation matrix with properties (1), (2), then there is a unitary $U: \mathcal{H}(K_S) \rightarrow \mathcal{X}$ so that $\begin{bmatrix} U & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}$.

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$$\begin{aligned} A^* : f(z) &\mapsto \frac{f(z) - f(0)}{\bar{z}}, & C^* : y &\mapsto \frac{S(z)^* - S(0)^*}{\bar{z}} y, \\ B : u &\mapsto (I_{\mathcal{U}} - S(z)^* S(0)) u, & D : u &\mapsto S(0) u. \end{aligned}$$

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- (1) \mathbf{U} is an isometry and (A, B) is a *controllable pair*, i.e., $\bigvee_{n \geq 0} \text{Ran} A^n B = \mathcal{H}(\tilde{K}_S)$.
- (2) We recover $S(z)$ as $S(z) = D + zC(I - zA)^{-1}B$.
- (3) If $\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} : (\mathcal{X} \oplus \mathcal{U}) \rightarrow (\mathcal{X} \oplus \mathcal{Y})$ is another colligation matrix with properties (1),(2), then it is unitarily equivalent to \mathbf{U} .

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Let $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and let $\mathcal{H}(\widehat{K}_S)$ be the RKHS with RK $\widehat{K}_S(z, \zeta)$. Define $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(\widehat{K}_S) \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}(\widehat{K}_S) \\ \mathcal{Y} \end{bmatrix}$ by

$$\begin{aligned} A : \begin{bmatrix} f(z) \\ g(z) \end{bmatrix} &\mapsto \begin{bmatrix} [f(z) - f(0)]/z \\ \bar{z}g(z) - S(z)^*f(0) \end{bmatrix}, & B : u &\mapsto \begin{bmatrix} \frac{S(z) - S(0)}{z}u \\ (I - S(z)^*S(0))u \end{bmatrix} \\ C : \begin{bmatrix} f(z) \\ g(z) \end{bmatrix} &\mapsto f(0), & D : u &\mapsto S(0)u. \end{aligned}$$

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(1) \mathbf{U} is unitary and *closely connected*:

$$\bigvee_{n \geq 0} \{\text{Ran } A^n B, \text{Ran } A^{*n} C^*\} = \mathcal{H}(\widehat{K}_S).$$

(2) We recover $S(z)$ as $S(z) = D + zC(I - zA)^{-1}B$.

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$$M_z^* : f(z) \rightarrow \frac{f(z) - f(0)}{z} \rightarrow \mathbf{M}_z^* = (M_{z_1}^*, \dots, M_{z_d}^*):$$

$$f(z) - f(0) = z_1(M_{z_1}^* f)(z) + \dots + z_d(M_{z_d}^* f)(z) \quad \forall f \in \mathcal{H}(k_d).$$

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$$K_S(z, \zeta) = \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - \langle z, \zeta \rangle} \quad \text{is positive on } \mathbb{B}^d.$$

Theorem:

If $S: \mathbb{B}^d \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$ belongs to $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, then there is a unitary colligation

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix} \quad (1)$$

so that $S(z)$ can be expressed as

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where $Z_{\mathcal{X}}(z) := \begin{bmatrix} z_1 I_{\mathcal{X}} & \dots & z_d I_{\mathcal{X}} \end{bmatrix}$.

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where $Z_{\mathcal{X}}(z) := \begin{bmatrix} z_1 I_{\mathcal{X}} & \dots & z_d I_{\mathcal{X}} \end{bmatrix}$.

Conversely, if \mathbf{U} of the form (1) is a contraction, then S of the form (2) belongs to $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ (Agler-McCarthy, Ball-Trent-Vinnikov).

Minimality conditions

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Equivalently, \mathbf{U} is weakly-coisometric if for $S(z) = D + C(I_{\mathcal{X}} - Z_{\mathcal{X}}(z)A)^{-1}Z_{\mathcal{X}}(z)B$,

$$\kappa_S(z, \zeta) := \frac{\|y - S(z)S(\zeta)^*\|}{1 - \langle z, \zeta \rangle} = C(I_{\mathcal{X}} - Z_{\mathcal{X}}(z)A)^{-1}(I - A^*Z_{\mathcal{X}}(\zeta)^*)^{-1}C^*.$$

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(3) The operators $C : \mathcal{H}(K_S) \rightarrow \mathcal{Y}$ and $D : \mathcal{U} \rightarrow \mathcal{Y}$ are given by

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$$A_i = U\tilde{A}_iU^*, \quad C = \tilde{C}U^*, \quad B_i = U\tilde{B}_i \quad (i = 1, \dots, d).$$

Nonuniqueness

By definition of K_S ,

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or, in the inner product form as

$$\begin{aligned} & \left\langle \begin{bmatrix} Z_{\mathcal{Y}}(\zeta)^* K_S(\cdot, \zeta) y \\ y \end{bmatrix}, \begin{bmatrix} Z_{\mathcal{Y}}(z)^* K_S(\cdot, z) y' \\ y' \end{bmatrix} \right\rangle_{\mathcal{H}(K_S)^d \oplus \mathcal{Y}} \\ &= \left\langle \begin{bmatrix} K_S(\cdot, \zeta) y \\ S(\zeta)^* y \end{bmatrix}, \begin{bmatrix} K_S(\cdot, z) y' \\ S(z)^* y' \end{bmatrix} \right\rangle_{\mathcal{H}(K_S) \oplus \mathcal{U}}. \end{aligned}$$

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Thus the map $V: \begin{bmatrix} Z_Y(\zeta)^* K_S(\cdot, \zeta) y \\ y \end{bmatrix} \rightarrow \begin{bmatrix} K_S(\cdot, \zeta) y \\ S(\zeta)^* y \end{bmatrix}$ extends to the isometry $V: \mathcal{D}_V \rightarrow \mathcal{R}_V$ where

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$$\mathcal{D}^\perp = \left\{ h = \begin{bmatrix} h_1 \\ \vdots \\ h_d \end{bmatrix} \in \mathcal{H}(K_S)^d : z_1 h_1(z) + \dots + z_d h_d(z) \equiv 0 \right\}.$$

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Thus the nonuniqueness is in $\left. \begin{bmatrix} A^* \\ B^* \end{bmatrix} \right|_{\mathcal{D}^\perp}$.

Commutative realizations

Let us call a realization

$$S(z) = D + C(I - z_1A_1 - \dots - z_dA_d)^{-1}(zB_1 + \dots + z_dB_d)$$

commutative if the tuple $\mathbf{A} = (A_1, \dots, A_d)$ is commutative.

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Theorem: A function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ admits a commutative weakly coisometric realization if and only if the space $\mathcal{H}(K_S)$ is M_z^* -invariant and

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In this case there exists a commutative **cfm** realization for S and for any such a realization, the state space operators are

$$A_j = M_{z_j}^*|_{\mathcal{H}(K_S)} \quad (i = 1, \dots, d).$$

Characteristic function of a commuting row contraction

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where

$$\begin{aligned} D_T &= (I_{\mathcal{X}^d} - T^*T)^{1/2}, & \mathcal{D}_T &= \overline{\text{Ran } D_T} \subset \mathcal{X}^d, \\ D_{T^*} &= (I_{\mathcal{X}} - TT^*)^{1/2}, & \mathcal{D}_{T^*} &= \overline{\text{Ran } D_{T^*}} \subset \mathcal{X}, \end{aligned}$$

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$$\mathbf{U}_{\mathbf{T}} = \begin{bmatrix} T^* & D_T \\ D_{T^*} & -T \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{D}_T \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}^d \\ \mathcal{D}_{T^*} \end{bmatrix}$$

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is unitary. Thus, $\theta_{\mathbf{T}}$ belongs to $\mathcal{S}(\mathcal{D}_T, \mathcal{D}_{T^*})$, admits a commutative unitary realization and is pure:

$$\|S(0)u\| = \|u\| \quad \text{for some } u \in \mathcal{U} \implies u = 0.$$

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Theorem: A function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ coincides with a characteristic function $\theta_{\mathbf{T}}$ of a commutative **c.n.c.** row contraction \mathbf{T} if and only if S is pure, the space $\mathcal{H}(K_S)$ is \mathbf{M}_z^* -invariant,

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and $\dim \text{Ker} A^*|_{\mathcal{D}^\perp} = \dim \mathcal{U}_S^0$, where

$$A = \begin{bmatrix} M_{z_1}^*|_{\mathcal{H}(K_S)} \\ \vdots \\ M_{z_d}^*|_{\mathcal{H}(K_S)} \end{bmatrix}, \quad \mathcal{U}_S^0 = \{u \in \mathcal{U} : S(z)u \equiv 0\}.$$

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Theorem: Two commutative **c.n.c.** row contractions T and R are unitarily equivalent if and only if their characteristic functions θ_T and θ_R coincide (Bhattacharyya-Eschmeier-Sarkar).

A contractive $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} =: \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix}$ is *weakly-coisometric* if \mathbf{U}^* is isometric on the subspace $\mathcal{D}_{C,A} \oplus \mathcal{Y}$ of $\mathcal{X}^d \oplus \mathcal{Y}$ where

$$\mathcal{D}_{C,A} := \bigvee_{\zeta \in \mathbb{B}^d, y \in \mathcal{Y}} Z_{\mathcal{X}}(\zeta)^*(I - A^*Z_{\mathcal{X}}(\zeta)^*)^{-1}C^*y \subset \mathcal{X}^d,$$

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and *weakly unitary* if it is weakly isometric and weakly coisometric.

Weakly isometric realizations

To construct a weakly isometric functional-model realization we need a positive kernel

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which is not positive. We have instead a positive kernel

$$\Phi = \begin{bmatrix} \Phi_{11} & \dots & \Phi_{1d} \\ \vdots & & \vdots \\ \Phi_{d1} & \dots & \Phi_{dd} \end{bmatrix} : \mathbb{B}^d \times \mathbb{B}^d \rightarrow \mathcal{L}(\mathcal{U}^d)$$

such that

$$h_{\mathcal{U}} - S(z)^* S(\zeta) = \sum_{j=1}^d \Phi_{jj}(z, \zeta) - \sum_{i,\ell=1}^d \bar{z}_i \zeta_{\ell} \Phi_{i\ell}(z, \zeta).$$

Weakly unitary realizations

If $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, then there exists a positive kernel

$$\mathbb{K}(z, \zeta) = \begin{bmatrix} K_S(z, \zeta) & \Psi_1(z, \zeta) & \dots & \Psi_d(z, \zeta) \\ \Psi_1(\zeta, z)^* & \Phi_{11}(z, \zeta) & \dots & \Phi_{1d}(z, \zeta) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_d(\zeta, z)^* & \Phi_{d1}(z, \zeta) & \dots & \Phi_{dd}(z, \zeta) \end{bmatrix}$$

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$$S(z) - S(\zeta) = \sum_{j=1}^d (z_j - \zeta_j) \Psi_j(z, \zeta).$$

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and let $\mathbb{T}(z, \zeta) := \begin{bmatrix} \mathbb{K}_1(z, \zeta) \\ \vdots \\ \mathbb{K}_d(z, \zeta) \end{bmatrix} : \mathbb{B}^d \times \mathbb{B}^d \rightarrow \mathcal{L}(\mathcal{U}, (\mathcal{Y} \oplus \mathcal{U}^d)^d).$

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$$\mathcal{R} = \bigvee \left\{ \mathbb{K}_0(\cdot, \zeta)y, \sum_{j=1}^d \zeta_j \mathbb{K}_j(\cdot, \zeta)u : \zeta \in \mathbb{B}^d, y \in \mathcal{Y}, u \in \mathcal{U} \right\} \subset \mathcal{H}(\mathbb{K}).$$

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Define two linear maps $\mathbf{s} : \mathcal{H}(\mathbb{K}) \rightarrow \mathcal{H}(K_S)$ and $\tilde{\mathbf{s}} : \mathcal{H}(\mathbb{K})^d \rightarrow \mathcal{H}(\Phi)$

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3. $C : f \rightarrow (\mathbf{s}f)(0)$, $B^* : g \rightarrow (\tilde{\mathbf{s}}g)(0)$ and $D : u \rightarrow S(0)u$.

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1. A colligation $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a **tcfm** colligation associated with \mathbb{K} if and only if

$$\mathbf{U}^* = \begin{bmatrix} X & 0 \\ 0 & V \end{bmatrix} : \begin{bmatrix} \mathcal{D}^\perp \\ \mathcal{D} \oplus \mathcal{Y} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}^\perp \\ \mathcal{R} \oplus \mathcal{U} \end{bmatrix}$$

where $X : \mathcal{D}^\perp \rightarrow \mathcal{R}^\perp$ is a contraction.

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2. Every **tcfm** colligation \mathbf{U} associated with \mathbb{K} is weakly unitary and closely connected and furthermore,
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$$S(z) = D + C(I - Z_{\mathcal{H}(\mathbb{K})}(z)A)^{-1}Z_{\mathcal{H}(\mathbb{K})}(z)B.$$
3. Any weakly unitary closely connected colligation $\tilde{\mathbf{U}}$ with the characteristic function equal S is unitarily equivalent to a **tcfm** colligation \mathbf{U} associated with some Agler decomposition \mathbb{K} for S .