Invariants for Semi-Fredholm Hilbert Modules

Shibananda Biswas

Indian Statistical Institute, India

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Notations

- Let $M_i : \mathcal{M} \to \mathcal{M}$ be the operator defined by $f \mapsto z_i \cdot f$, where $z_i \cdot f$ is the module multiplication.
- Let $D_{M^*} : \mathcal{M} \to \mathcal{M} \oplus \cdots \oplus \mathcal{M}$ be the *m*-tuple operators $f \mapsto \big(M_1^* f, \ldots, M_m^* f\big)$.
- $\ker D_{(M-w)^*} = \cap_{j=1}^m \ker(M_j - w_j)^*$ for $w \in \mathbb{C}^m$.

Definition

A Hilbert module $\mathcal{M}$ over the polynomial ring $\mathcal{C}_m; = \mathbb{C}[z_1, \ldots, z_m]$ is said to be in the class $\mathrm{B}_n(\Omega)$ if

- The range of $D_{(M-w)^*}$ is closed in $\mathcal{M} \oplus \cdots \oplus \mathcal{M}$ for all $w \in \Omega$;
- $\mathrm{span}_{w\in\Omega} \ker D_{(M-w)^*}$ is dense in $\mathcal{M}$; and
- $\dim \ker D_{(M-w)^*} = n$ for all $w \in \Omega$.

Example of modules in $\mathrm{B}_n(\Omega)$

Examples of modules in $\mathrm{B}_1(\mathbb{D}^n)$:

- Hardy module
- Bergmann module
- Dirichlet module.

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Equivalence of modules in $\mathrm{B}_n(\Omega)$

$$E_{\mathcal{M}} := \{(w, f) \in \Omega \times \mathcal{M} : f \in \ker D_{(M-w)^*}\} \text{ and } \pi(w, f) = w$$

defines a holomorphic Hermitian vector bundle on the open set Ω.

Theorem (Cowen - Douglas)

Two Hilbert modules $\mathcal{M}$ and $\tilde{\mathcal{M}}$ in $\mathrm{B}_n(\Omega)$ iff the vector bundles $E_{\mathcal{M}}$ and $E_{\tilde{\mathcal{M}}}$ are equivalent as holomorphic Hermitian vector bundle.

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Model

Any Hilbert module $\mathcal{M}$ in $\mathrm{B}_n(\Omega)$ is isomorphic to a Hilbert module $\mathcal{M}_\Gamma \subseteq \mathcal{O}(\Omega)$ possessing a reproducing kernel on Ω.

Conversely, the adjoint of the m-tuple of multiplication operators on the reproducing kernel Hilbert space associated with a kernel K on Ω belongs to $\mathrm{B}_n(\Omega^*)$ if certain additional conditions are imposed on K.

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Natural example of module fails to be in $\mathrm{B}_1(\Omega)$

Let $H^2_0(\mathbb{D}^2) := \{f \in H^2(\mathbb{D}^2) : f(0) = 0\}$.

$$\dim \ker D_{(\mathbf{M}-w)^*} = \dim H^2_0(\mathbb{D}^2) \otimes_{\mathcal{C}_2} \mathbb{C}_w = \begin{cases} 1 & \text{if } w \neq (0,0) \\ 2 & \text{if } w = (0,0). \end{cases}$$

$\mathbb{C}_w$ is the one dimensional module over the polynomial ring $\mathcal{C}_2$, where the module action is given by the map $(f, \lambda) \mapsto f(w)\lambda$ for $f \in \mathcal{C}_2$ and $\lambda \in \mathbb{C}_w \cong \mathbb{C}$.

We consider the class of Hilbert modules where the the dimension of the joint kernel is no longer assumed to be constant.

Definition

$\mathcal{M} \in \mathfrak{B}_1(\Omega)$ iff

- $\mathcal{M}$ is a submodule of $\mathcal{H}$ for some $\mathcal{H} \in \mathrm{B}_1(\Omega)$ and
- $\dim \ker D_{(\mathbf{M}-w)^*} < \infty$ for all $w \in \Omega\}$

Clearly Hilbert modules in $\mathfrak{B}_1(\Omega)$

- possess a reproducing kernel K (we don't rule out the possibility: $K(w, w) = 0$ for w in some closed subset X of Ω) and
- the range of $D_{(\mathbf{M}-w)^*}$ is closed.

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Relation between $\mathfrak{B}_1(\Omega)$ and $\mathrm{B}_1(\Omega)$

The following Lemma isolates a very large class of elements from $\mathfrak{B}_1(\Omega)$ which belong to $\mathrm{B}_1(\Omega_0)$ for some open subset $\Omega_0 \subseteq \Omega$.

Lemma

Suppose $\mathcal{M} \in \mathfrak{B}_1(\Omega)$ is the closure of a polynomial ideal $\mathcal{I}$. Then $\mathcal{M}$ is in $\mathrm{B}_1(\Omega)$ if the ideal $\mathcal{I}$ is singly generated while if it is generated by the polynomials $p_1, p_2, \ldots, p_t$, then $\mathcal{M}$ is in $\mathrm{B}_1(\Omega \setminus X)$ for $X = \{z : p_1(z) = \ldots = p_t(z) = 0\}$.

Definition

- Analogous to the correspondence of a vector bundle with a locally free sheaf, we construct a sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ for the Hilbert module $\mathcal{M}$.

- The sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ is the subsheaf of the sheaf of holomorphic functions $\mathcal{O}(\Omega)$ whose stalk at $w \in \Omega$ is

$$\{(f_1)_w\mathcal{O}_w + \cdots + (f_n)_w\mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}\}$$

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A property of $\mathcal{S}^{\mathcal{M}}(\Omega)$

Proposition

For any Hilbert module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$, the sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ is coherent analytic.

Key ingredient in the proof:

- Noether's stationary lemma.

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The decomposition theorem

Theorem

Suppose g_i^0, $1 \leq i \leq d$, be a minimal set of generators for the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$. Then

- *there exists a open neighborhood Ω_0 of w_0 such that*

  $$K(\cdot, w) := K_w = g_1^0(w)K_w^{(1)} + \cdots + g_n^0(w)K_w^{(d)},\ w \in \Omega_0$$

  for some choice of anti-holomorphic functions $K^{(1)}, \ldots, K^{(d)} : \Omega_0 \to \mathcal{M}$,
- *the vectors $K_w^{(i)}$, $1 \leq i \leq d$, are linearly independent in $\mathcal{M}$ for w in some small neighborhood of w_0,*

Theorem continued

Theorem (contd.)

- *the vectors* $\{K_{w_0}^{(i)} \mid 1 \leq i \leq d\}$ *are uniquely determined by these generators* $g_1^0, \ldots, g_d^0$,
- *the linear span of the set of vectors* $\{K_{w_0}^{(i)} \mid 1 \leq i \leq d\}$ *in* $\mathcal{M}$ *is independent of the generators* $g_1^0, \ldots, g_d^0$, *and*
- *the vectors* $K_{w_0}^{(i)}$, $1 \leq i \leq d$, *are eigenvectors for the adjoint of the action of* $\mathcal{C}_m$ *on the Hilbert module* $\mathcal{M}$ *at* w_0.

Outline of the proof of the Theorem

Key ingredients:

- Every function in a submodule of $\mathcal{O}_{w_0}$ decomposes in terms of its generator on a small neighbourhood of w_0, with coefficients satisfying some norm bounds in a even smaller compact neighbourhood of 0.
- $\mathcal{O}_{w_0}$ is a local ring and the Nakayama's lemma

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An Inequality

The following is evident from previous theorem:

$$\begin{aligned}\dim\ker D_{(\mathbf{M}-w_0)^*} &\geq \sharp\{\textit{minimal generators for } S_{w_0}^{\mathcal{M}}\}\\ &\geq \dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}(\mathcal{O}_{w_0})\mathcal{S}_{w_0}^{\mathcal{M}}.\end{aligned}$$

Analytic Hilbert modules

Definition

A Hilbert module $\mathcal{M}$ over the polynomial ring $\mathcal{C}_m$ is said to be an *analytic Hilbert module* if we assume that

- it consists of holomorphic functions on a bounded domain $\Omega \subseteq \mathbb{C}^m$ and possesses a reproducing kernel K,
- the polynomial ring $\mathcal{C}_m$ is dense in it,
- the set of virtual points, which is $\{w \in \mathbb{C}^m : p \mapsto p(w),\ p \in \mathcal{C}_m \text{ is continuous}\}$, is Ω.

Class of examples where equality holds

Proposition

Let $\mathcal{M} = [\mathcal{I}]$ be a submodule of an analytic Hilbert module over $\mathcal{C}_m$, where $\mathcal{I}$ is an ideal in the polynomial ring $\mathcal{C}_m$. Then

$$\sharp\{\textit{minimal set of generators for } \mathcal{S}^{\mathcal{M}}_{w_0}\} = \dim\ker D_{(\mathbf{M}-w_0)^*}.$$

Outline of the proof of the Proposition

Key ingredient:

- $\dim \ker D_{(\mathbf{M}-w_0)^*} = \dim \mathcal{I}/\mathfrak{m}_{w_0}\mathcal{I}.$

Main step of the proof:

- To show that a map

$$\mathcal{I}/\mathfrak{m}_{w_0}\mathcal{I} \longrightarrow \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}(\mathcal{O}_{w_0})\mathcal{S}^{\mathcal{M}}_{w_0},$$

is one-one, where $\mathfrak{m}_{w_0}$ is the maximal ideal of $\mathcal{C}_m$ at w_0.

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An important corollary

Corollary

If $\mathcal{M} = [\mathcal{I}]$ be a submodule of an analytic Hilbert module over $\mathcal{C}_m$, where $\mathcal{I}$ is an ideal in the polynomial ring $\mathcal{C}_m$ and $w_0 \in V(\mathcal{I})$ is a smooth point, then

$$\dim \ker D_{(\mathbf{M}-w_0)^*} = \begin{cases} 1 & \text{for } w_0 \notin V(\mathcal{I}) \cap \Omega; \\ \text{codimension of } V(\mathcal{I}) & \text{for } w_0 \in V(\mathcal{I}) \cap \Omega. \end{cases}$$

Constancy of dimension in a neighbourhood

Let $\mathbb{P}_0$ be the orthogonal projection onto $\mathrm{ran} D_{(\mathbf{M}-w_0)^*}$.

Lemma

The dimension of $\ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}$ *is constant in a suitably small neighbourhood of* $w_0 \in \Omega$*, say* Ω_0*.*

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Construction of a vector bundle

$$\mathcal{P}^{\mathcal{M}}_{w_0} := \{(w,f) \in \Omega \times \mathcal{M} : f \in \ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}\} \text{ and } \pi(w,f) = w$$

defines a holomorphic Hermitian vector bundle on the open set Ω_0.

Theorem

If any two Hilbert modules $\mathcal{M}$ and $\tilde{\mathcal{M}}$ from $\mathfrak{B}_1(\Omega)$ are isomorphic via an unitary module map, then the corresponding holomorphic Hermitian vector bundles $\mathcal{P}^{\mathcal{M}}_{w_0}$ and $\mathcal{P}^{\tilde{\mathcal{M}}}_{w_0}$ on Ω_0 are equivalent.

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Outline of the proof of the Theorem

Key ingredients:

- Existence of the operator $R_{\mathbf{M}}(w)$ such that the following holds on Ω

$$\begin{aligned} R_{\mathbf{M}}(w) D_{(\mathbf{M}-w)^*} &= I - P_{\ker D_{(\mathbf{M}-w)^*}} \\ D_{(\mathbf{M}-w)^*} R_{\mathbf{M}}(w) &= P_{\operatorname{ran} D_{(\mathbf{M}-w)^*}}, \end{aligned}$$

- Construction of the operator

$$P(\bar{w}, \bar{w}_0) = I - \{I - R_{\mathbf{M}}(w_0) D_{\bar{w}-\bar{w}_0}\}^{-1} R_{\mathbf{M}}(w_0) D_{(\mathbf{M}-w)^*},$$

for $w \in B(w_0; \| R(w_0) \|^{-1})$.

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for $w \in B(w_0; \| R(w_0) \|^{-1})$.

Calculation of the invariant for a class of Hilbert module

For $\lambda, \mu > 0$, $H_0^{(\lambda,\mu)}(\mathbb{D}^2) := \{f \in H^{(\lambda,\mu)}(\mathbb{D}^2) : f(0,0) = 0\}$,

where $H^{(\lambda,\mu)}(\mathbb{D}^2)$ be the reproducing kernel Hilbert space on the bi-disc determined by the positive definite kernel

$$K^{(\lambda,\mu)}(z,w) = \frac{1}{(1 - z_1\bar{w}_1)^\lambda(1 - z_2\bar{w}_2)^\mu},\ z,w \in \mathbb{D}^2.$$

Calculation of the invariant for a class of Hilbert module

The normalized metric $h_0(w,w)$, which is real analytic, is of the form

$$h_0(w,w) = I + \begin{pmatrix} \frac{\lambda+1}{2}|w_1|^2 + \frac{\lambda^2\mu}{(\lambda+\mu)^2}|w_2|^2 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_1\bar{w}_2 \\ \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_2\bar{w}_1 & \frac{\lambda\mu^2}{(\lambda+\mu)^2}|w_1|^2 + \frac{\mu+1}{2}|w_2|^2 \end{pmatrix}$$

$$+O(|w|^3),$$

where $O(|w|^3)_{i,j}$ is of degree ≥ 3.

Calculation of the invariant for a class of Hilbert module

The curvature for $\mathcal{P}$ at $(0,0)$ is given by the 2×2 matrices

$$\begin{pmatrix} \frac{\lambda+1}{2} & 0 \\ 0 & \frac{\lambda\mu^2}{(\lambda+\mu)^2} \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 & 0 \end{pmatrix},$$

$$\text{and } \begin{pmatrix} \frac{\lambda^2\mu}{(\lambda+\mu)^2} & 0 \\ 0 & \frac{\mu+1}{2} \end{pmatrix}.$$

Calculation of the invariant for a class of Hilbert module

Lemma

$H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ are equivalent if and only if $\lambda = \lambda'$ and $\mu = \mu'$.

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THANK YOU.