Standard Hilbert modules and the K-homology of algebraic varieties

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Preview

We give a birds-eye survey of the problem of constructing explicit *examples* in multivariable operator theory, focusing on unsolved problems and conjectures. For details, see

- TAMS (2007) v. 359, pp. 6027–6055.

  - Review of some background results of Hilbert on what might be called *multivariable linear algebra.*
  - The issue: How should one *construct* the Hilbert space counterparts of projective algebraic varieties and related objects (like vector bundles or sheaves over varieties)? More precisely, how does one construct the K-homology classes of algebraic varieties?

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Multivariable linear algebra

V: complex vector space (typically infinite-dimensional), $T_1, \ldots, T_d$ commuting linear operators on V.

Regard V as a module over $\mathbb{C}[z_1, \ldots, z_d]$:

$$f \cdot \xi = f(T_1, \ldots, T_d)\xi, \qquad f \in C[z_1, \ldots, z_k], \quad \xi \in V.$$

Finitely generated: there exist $\xi_1, \ldots, \xi_r \in V$ such that

$$V = \{f_1 \cdot \xi_1 + \cdots + f_r \cdot \xi_r : f_k \in \mathbb{C}[z_1, \ldots, z_d]\}.$$

If we identify r-tuples of polynomials in $\mathbb{C}[z_1, \ldots, z_d]$ in the natural way with elements of $C[z_1, \ldots, z_d] \otimes \mathbb{C}^r$, then we can define a surjective homomorphism of modules

$$\mathbb{C}[z_1, \ldots, z_d] \otimes \mathbb{C}^r \to V \to 0$$

by sending an r-tuple of polynomials $(f_1, \ldots, f_r)$ to the vector

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Typically, this map has nontrivial kernel K

$$0 \to K \to \mathbb{C}[z_1, \ldots, z_d] \otimes \mathbb{C}^r \to V \to 0.$$

However, Hilbert's basis theorem implies that K is finitely generated too. So we can choose $\eta_1, \ldots, \eta_s \in K$ such that

$$K = \{f_1 \cdot \eta_1 + \cdots + f_s \cdot \eta_s : f_k \in \mathbb{C}[z_1, \ldots, z_d]\},$$

and repeat the procedure to get a longer exact sequence

$$\mathbb{C}[z_1, \ldots, z_k] \otimes \mathbb{C}^s \to \mathbb{C}[z_1, \ldots, z_k] \otimes \mathbb{C}^r \to V \to 0.$$

If the map on the left has nonzero kernel, we continue (perhaps forever) to obtain a *free resolution* of V – an exact sequence of finitely generated free modules (i.e., modules of the form $\mathbb{C}[z_1, \ldots, z_d] \otimes \mathbb{C}^k$) that terminates in the original module V.

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Hilbert's syzygy theorem

Theorem (Math. Ann. (1893))

Every finitely generated $\mathbb{C}[z_1, \ldots, z_d]$-module V has a finite free resolution of length at most d in the sense that there are integers $r_1, \ldots, r_n \geq 0$, $n \leq d$, such that

$$0 \to \mathbb{C}[z_1, \ldots, z_d] \otimes \mathbb{C}^{r_n} \to \cdots \to \mathbb{C}[z_1, \ldots, z_d] \otimes \mathbb{C}^{r_1} \to V \to 0$$

is exact.

- Every free resolution can be reduced to a *minimal* one.
- All minimal free resolutions are isomorphic.

- Application: One can calculate the *Euler characteristic* of V by using *any* free resolution of V:

$$\chi(V) = r_1 - r_2 + r_3 - r_4 \pm \cdots + (-1)^n r_n.$$

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David Hilbert ca 1900

Graded modules over $\mathbb{C}[z_1, \ldots, z_d]$

The polynomials form a *graded* algebra,

$$\mathbb{C}[z_1, \ldots, z_d] = \mathcal{P}_0 \dot{+} \mathcal{P}_1 \dot{+} \mathcal{P}_2 \dot{+} \cdots$$

where $\mathcal{P}_n =$ homogeneous polynomials of degree n, and one has $\mathcal{P}_m \cdot \mathcal{P}_n \subseteq \mathcal{P}_{m+n}$. To lighten notation, we write $\mathcal{A}_d$ instead of $\mathbb{C}[z_1, \ldots, z_d]$, or simply $\mathcal{A}$ when the dimension d is understood.

An $\mathcal{A}$-module V is said to be *graded* when

$$V = V_0 \dot{+} V_1 \dot{+} V_2 \dot{+} \cdots$$

where $z_j V_k \subseteq V_{k+1}$ for all $1 \leq k \leq d$, $k = 0, 1, 2, \ldots$.
Example: The free module of rank r, namely $\mathcal{A} \otimes \mathbb{C}^r$, “is" the space of vector-valued polynomials (taking values in $\mathbb{C}^r$)

$$A \otimes \mathbb{C}^r = F_0 \dot{+} F_1 \dot{+} F_2 \dot{+} \cdots$$

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Graded $\mathcal{A}$-modules (cont.)

There is a fairly obvious “graded" variant of the syzygy theorem.

In particular, the most general finitely generated graded module V over $\mathcal{A}_d$ can be constructed by a two-step procedure:

• Step 1: Choose a graded submodule $M = M_0 \dotplus M_1 \dotplus M_2 \dotplus \cdots$ of the graded free module of rank r

$$F = \mathcal{A}_d \otimes \mathbb{C}^r = F_0 \dotplus F_1 \dotplus F_2 \dotplus \cdots$$

• Step 2: Form the graded quotient module

$$V = F/M = (F_0/M_0) \dotplus (F_1/M_1) \dotplus \cdots .$$

Such modules can represent (the algebras of polynomials on) projective algebraic varieties, or (the sections of) vector bundles or sheaves over projective algebraic varieties.

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Can we do this in Hilbert space?

In more concrete terms, this algebraic construction gives rise to d-tuples of commuting operators $T_1, \ldots, T_d$ that satisfy systems of equations of the form

$$f_k(T_1, \ldots, T_d) = 0, \qquad k = 1, \ldots, s,$$

where $f_1, \ldots, f_s$ is a finite set of homogeneous polynomials (perhaps of different degrees).

The set X of common zeros of $\{f_1, \ldots, f_k\}$ is a projective algebraic variety.

We want to construct Hilbert space counterparts of such d-tuples so as to obtain K-homology classes of X in concrete terms (as well as the accompanying index theorems).

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What doesn't work, and why not?

As a simple example, consider the problem of constructing commuting triples of operators $X, Y, Z \in \mathcal{B}(H)$ that satisfy

$$X^n + Y^n = Z^n$$

for some $n = 2, 3, \ldots$.

E.g., one can start with a pair of commuting operators X, Y and look for an n th root Z of $X^n + Y^n$. Unfortunately, many operators don't have n th roots (Example: the unilateral shift).

So *ad hoc* methods fail. Instead, we need to deal directly with *quotients* of Hilbert modules such as H/M where

- H is a "free" Hilbert module in three variables X, Y, Z, and
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Some terminology

- Hilbert module over $\mathcal{A}_d$: A Hilbert space H endowed with commuting operators $T_1, \dots, T_d \in \mathcal{B}(H)$ for which

$$f \cdot \xi = f(T_1, \dots, T_d)\xi, \qquad f \in \mathcal{A}_d, \quad \xi \in H.$$

- Grading of H: An $\perp$ decomposition $H = H_0 \oplus H_1 \oplus H_2 \oplus \cdots$ for which $T_j H_k \subseteq H_{k+1}$ for all $1 \leq j \leq d$, $k = 0, 1, 2, \dots$.

- Obvious meaning of finitely generated Hilbert module.

- The C^*-algebra of a Hilbert module H over $\mathcal{A}_d$: The unital C^*-algebra generated by the "coordinate" operators $T_1, \dots, T_d$

$$C^*(H) = C^*\{\mathbf{1}, T_1, \dots, T_d\}.$$

- H is said to be essentially normal if $C^*(H)$ is commutative modulo compacts. H is p-essentially normal (for $1 \leq p \leq \infty$) if the cross commutators $T_j^* T_k - T_k T_j^*$ all belong to $\mathcal{L}^p$.

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• Obvious meaning of finitely generated Hilbert module.

• The C^*-algebra of a Hilbert module H over $\mathcal{A}_d$: The unital C^*-algebra generated by the “coordinate" operators $T_1, \dots, T_d$

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What are free Hilbert modules?

To get started, what should be the Hilbert space counterparts of the free module of rank 1

$$V = \mathbb{C}[z_1, \dots, z_d]?$$

These will be called *graded completions* (of $\mathcal{A} = \mathbb{C}[z_1, \dots, z_d]$), and they are defined as follows....

• A *graded inner product* is an inner product $\langle \cdot, \cdot \rangle$ on

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with the following two properties:

(i): $\mathcal{P}_m \perp \mathcal{P}_n$ if $m \neq n$.

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Graded completions (of $\mathcal{A}_d$)

The completion G of $\mathcal{A}_d$ in $\langle \cdot, \cdot \rangle$ is obviously a graded Hilbert module (with a single generator - the constant polynomial 1).

(iii): If, in addition to (i) and (ii), the subspace

$$Z_1G + Z_2G + \cdots + Z_dG$$

is closed, then G is called a *graded completion* of $\mathcal{A}_d$.

- Something to keep in mind: There is only one free module of rank 1 in d-dimensional linear algebra. But in the category of Hilbert modules, there are uncountably many inequivalent graded completions of $\mathcal{A}_d$, with vastly different properties.

(Examples momentarily)

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Basic properties of all graded completions

Number operator N: Unbounded positive operator, defined by

$$Nf = n \cdot f, \qquad f \in \mathcal{P}_n, \quad n = 0, 1, 2, \dots .$$

It satisfies

$$\operatorname{trace}((\mathbf{1} + N)^{-p}) < \infty, \qquad \forall p > d.$$

Up to unitary equivalence, all graded completions have the “same" number operator.

They also share an irreducibility property:

Proposition: Every graded completion G is irreducible, and in fact $\mathcal{K} \subseteq C^*(G)$.

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A standard Hilbert module is finite-multiplicity version of a graded completion G - a Hilbert module of the form $S = G \otimes \mathbb{C}^r$

$$f \cdot (g \otimes \zeta) = (f \cdot g) \otimes \zeta, \qquad g \in G, \quad \zeta \in \mathbb{C}^r,$$

where $r = 1, 2, \dots$.

We focus on graded quotients of standard Hilbert modules: i.e.,

$$H = S/M$$

where S is standard and $M \subseteq S$ is a *graded* submodule.

• Key issue: Is $H = S/M$ essentially normal? Equivalently, do we have an exact sequence of C^*-algebras

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All of these are p-essentially normal for every $p > d$:

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Some other essentially normal graded completions:

• $\Omega \subseteq \mathbb{C}^d$ a bounded strongly pseudoconvex domain with smooth boundary. The Bergman module of Ω is essentially normal (PDE book of Michael Taylor, v. 2, Prop. 7.3 Chap. 12).

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Vladimir Aleksandrovich Fock 1898–1974

Hardy (ca 1920) and Bergman (ca 1955)

And let’s not forget Michael....

Bad ones

- Douglas and Howe observed that, among other things, the Bergman modules of polydisks are *not* essentially normal.

For the Bergman module H of the bi-disk $D \times D$,
$C^*(H) = \mathcal{T} \otimes \mathcal{T}$, where $\mathcal{T}$ = Toeplitz C^*-algebra

$$\mathcal{K}(H) = \mathcal{K} \otimes \mathcal{K} \subseteq \mathcal{K} \otimes \mathcal{T} \subseteq \mathcal{K} \otimes \mathcal{T} + \mathcal{T} \oplus \mathcal{K} \subseteq C^*(H)$$

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Worse ones

• It gets worse: (Upmeier) The modules of many symmetric domains have type I C^*-algebras with *arbitrarily long* Kaplansky composition series.

On the other hand, they are still type I. More importantly, their index theory is nice.

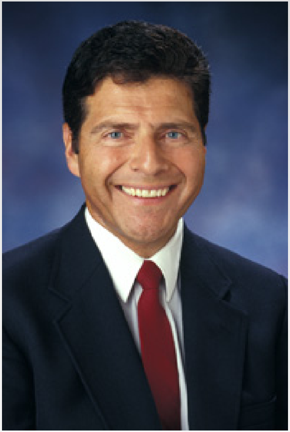
Harald ca. 1977

And still worse

Fix $a, b \in (0,1)$. Curto and Muhly (1985) showed that the C^*-algebra of the Bergman module of the “iron cross"

$$\Omega_{a,b} = \{(z,w) : |z| < a, |w| < 1\} \cup \{(z,w) : |z| < 1, |w| < b\}$$

is type $I \iff \log a / \log b$ is rational.



Submodules and quotients

Let S be a standard HIlbert module. We are interested in graded quotients of S, especially essentially normal ones.

Theorem: Let S be an essentially normal standard Hilbert module S and let $M \subseteq S$ be a graded submodule. TFAE:

1. S/M is essentially normal.
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• Similar result for p-essentially normal quotients, $p > d$.

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Structure of graded submodules

Fix $S = G \otimes \mathbb{C}^r$ standard Hilbert module over $\mathcal{A} = \mathcal{A}_d$, assumed to be p-essentially normal for some $d < p \leq \infty$.

Fix an integer $s \geq 0$ (the degree), and pick a linear space $E \subseteq \mathcal{P}_s$ of homogeneous polynomials of degree s. Then

$$M = [\mathcal{A}E] = E \oplus [z_j E : 1 \leq j \leq d] \oplus [z_i z_j E : 1 \leq i, j \leq d] \oplus \cdots$$

is a graded submodule; s is called the degree of M.

Proposition: Every graded submodule of S is a finite rank perturbation of one of this form, for some $s \geq 0$.

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A Conjecture

We now restrict attention to standard Hilbert modules $S = G \otimes \mathbb{C}^r$ based on the symmetric Fock graded completion G.

• **Basic Conjecture:** Every graded submodule of such an S is p-essentially normal for every $p > d$, that is

$$T_j^* T_k - T_k T_j^* \in \mathcal{L}^p, \qquad \forall p > d.$$

Significant consequences include

- Index formula for the curvature invariant.
- Homotopy invariance of the curvature invariant.
- The Koszul complex of every “universal" d-contraction has finite dimensional cohomology.
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Positive evidence: special cases

- True if M is generated by *monomials* (A. JOT 2005). That is, if $M \subseteq G \otimes \mathbb{C}^r$ is generated by vector-valued polynomials

$$f(z_1, \ldots, z_d) = z_1^{n_1} z_2^{n_2} \cdots z_d^{n_d} \otimes \zeta$$

This “monomials" result persists for *certain* other graded completions (Douglas, JOT 2005).

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- True in a variety of other cases with special features that make it possible to decide.

- True for *all* graded submodules of $G \otimes \mathbb{C}^r$ in dimension $d = 2$ or $d = 3$ (Guo-Wang, 2008). The 4-variable case remains open!

Positive evidence: special cases

- True if M is generated by *monomials* (A. JOT 2005). That is, if $M \subseteq G \otimes \mathbb{C}^r$ is generated by vector-valued polynomials

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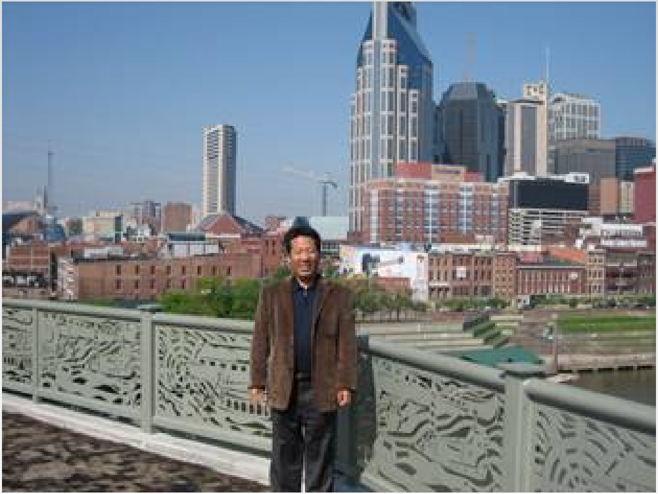

Kunyu Guo and Shanghai 2008

The case degree $= 1$ is sufficient

Linearization: (A. 2007) In any dimension d, let S be a standard Hilbert module based on an arbitrary p-essentially normal graded completion (for fixed $p > d$).

• If every *degree one* graded submodule of $S = G \otimes \mathbb{C}^r$ is p-essentially normal, then every graded submodule of S is p-ess. normal.

Note that the degree 1 submodules of $G \otimes \mathbb{C}^r$ are the invariant subspaces generated by sets of polynomials of the form

$$f(z_1, \dots, z_d) = z_1\zeta_1 + z_2\zeta_2 + \cdots + z_d\zeta_d$$

where $\zeta_1, \dots, \zeta_d$ are vectors in $\mathbb{C}^r$.

• **Conclusion:** It is not the *nonlinearity* of the generating polynomials that makes trouble. All the trouble is caused by “bad angles" in the space of *linear vector-valued polynomials*.

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Th - Th - Th - That’s all folks!