Carleson Measures for Besov-Sobolev Spaces and Non-Homogeneous Harmonic Analysis

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This talk is based on joint work with:

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Michigan State University
USA

Talk Outline

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- Main Results and Sketch of Proof

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- Further Results

Besov-Sobolev Spaces

- The space $B_2^\sigma(\mathbb{B}_n)$ is the collection of holomorphic functions f on the unit ball $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$ such that

$$\left\{\sum_{k=0}^{m-1}\left|f^{(k)}(0)\right|^2 + \int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z)\right|^2 d\lambda_n(z)\right\}^{\frac{1}{2}} < \infty,$$

where $d\lambda_n(z) = \left(1-|z|^2\right)^{-n-1} dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m+\sigma > \frac{n}{2}$.

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  - $\sigma = \frac{n+1}{2}$: Bergman Space; $k_\lambda^{\frac{n+1}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$

Carleson Measures for Besov-Sobolev Spaces

Definition

A non-negative Borel measure μ is a $B_2^\sigma(\mathbb{B}_n)$-Carleson measure if

$$\int_{\mathbb{B}_n} |f(z)|^2 d\mu(z) \leq C(\mu)^2 \|f\|_{B_2^\sigma(\mathbb{B}_n)}^2 \quad \forall f \in B_2^\sigma(\mathbb{B}_n).$$

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As is well-known Carleson measures play an important role in both the function theory of the space (e.g., characterization of the multipliers for the space and the interpolating sequences) and more generally play a predominant role in harmonic analysis.

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The definition is in terms of function theoretic information, and it would be more useful to have a “testable” condition to check.

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Testing on the reproducing kernel k_λ^σ we always have a necessary geometric condition for the measure μ to be Carleson:

$$\mu\left(T(B_r(\xi))\right) \lesssim r^{2\sigma} \quad \forall \xi \in \partial\mathbb{B}_n,\ r > 0$$

Here $T(B_r(\xi))$ is the tent over the ball or radius $r > 0$ in the boundary $\partial\mathbb{B}_n$

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- If $n = 1$, then the results can be expressed in terms of capacity conditions. More precisely,

$$\mu\left(T(\Omega)\right) \lesssim \mathrm{Cap}_\sigma\left(\Omega\right) \quad \forall \text{open}\, \Omega \subset \mathbb{T}.$$

  See for example Stegenga, Maz'ya, Verbitsky, Carleson.

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Question (Main Problem: Characterization in the Difficult Range)

Characterize the Carleson measures when $\frac{1}{2} < \sigma < \frac{n}{2}$.

Operator Theoretic Characterization of Carleson Measures

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We can give a characterization of Carleson measures for the space $\mathcal{J}$ in terms of information about the boundedness of a certain linear operator related to the reproducing kernel j_x.

Operator Theoretic Characterization of Carleson Measures

Statement and Proof

Proposition

A measure μ *is a* $\mathcal{J}$*-Carleson measure if and only if the linear map*

$$f(z) \rightarrow T(f)(z) = \int_X \mathrm{Re}\, j_x(z) f(x) d\mu(x)$$

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Proof:

The inclusion map ι is bounded from $\mathcal{J}$ to $L^2(X;\mu)$ if and only if the adjoint map ι^* is bounded from $L^2(X;\mu)$ to $\mathcal{J}$, namely,

$$\|\iota^* f\|_{\mathcal{J}}^2 = \langle \iota^* f, \iota^* f \rangle_{\mathcal{J}} \leq C \|f\|_{L^2(X;\mu)}^2, \quad \forall f \in L^2(X;\mu).$$

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Proof of Proposition

For an $x \in X$ we have

$$\begin{aligned} \iota^* f(x) = \langle \iota^* f, j_x \rangle_{\mathcal{J}} &= \langle f, \iota j_x \rangle_{L^2(X:\mu)} \\ &= \int_X f(w) \overline{j_x(w)} d\mu(w) \\ &= \int_X f(w) j_w(x) d\mu(w). \end{aligned}$$

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Using this computation, we obtain that

$$\begin{aligned} \| \iota^* f \|_{\mathcal{J}}^2 &= \langle \iota^* f, \iota^* f \rangle_{\mathcal{J}} \\ &= \left\langle \int_X j_w f(w) d\mu(w), \int_X j_{w'} f(w') d\mu(w') \right\rangle_{\mathcal{J}} \end{aligned}$$

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Those computations then give

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$$\|\iota^* f\|_{\mathcal{J}}^2 = \int_X \int_X \operatorname{Re} j_w(w') f(w) f(w') d\mu(w) d\mu(w') = \langle Tf, f\rangle_{L^2(X;\mu)}.$$

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But the last quantity satisfies the required estimates exactly when the operator T is bounded. □

Connections to Calderón-Zygmund Operators

When we apply this proposition to the spaces $B_2^\sigma(\mathbb{B}_n)$ this suggests that we study the operator

$$T_{\mu,2\sigma}(f)(z) = \int_{\mathbb{B}_n} \mathrm{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) f(w)d\mu(w) : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$$

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Theorem (David and Journé)

If T is a Calderón-Zygmund operator then $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ if and only if $T(1), T^*(1) \in BMO(\mathbb{R}^n)$ and T is weak bounded.

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Note that $T(1) \in BMO(\mathbb{R}^n)$ if and only if $\|T1_Q\|^2_{L^2(\mathbb{R}^n)} \lesssim |Q|$ for all Q.

Connections to Calderón-Zygmund Operators

When we apply this proposition to the spaces $B_2^\sigma(\mathbb{B}_n)$ this suggests that we study the operator

$$T_{\mu,2\sigma}(f)(z) = \int_{\mathbb{B}_n} \mathrm{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) f(w)d\mu(w) : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$$

and find some conditions that will let us determine when it is bounded. Recall the following theorem of David and Journé:

Theorem (David and Journé)

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Idea: Try to use T(1) to characterize the boundedness of

$$T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu).$$

Calderón–Zygmund Estimates for $T_{\mu,2\sigma}$

If we define

$$\Delta(z,w) := \begin{cases} \left||z|-|w|\right| + \left|1-\frac{z\overline{w}}{|z||w|}\right| & : \quad z,w\in\mathbb{B}_n\setminus\{0\} \\ |z|+|w| & : \quad \text{otherwise.} \end{cases}$$

Then Δ is a pseudo-metric and makes the ball into a space of homogeneous type.

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A computation demonstrates that the kernel of $T_{\mu,2\sigma}$ satisfies the following estimates:

$$|K_{2\sigma}(z,w)| \lesssim \frac{1}{\Delta(z,w)^{2\sigma}} \quad \forall z,w \in \mathbb{B}_n;$$

If $\Delta(\zeta,w) < \frac{1}{2}\Delta(z,w)$ then

$$|K_{2\sigma}(\zeta,w) - K_{2\sigma}(z,w)| \lesssim \frac{\Delta(\zeta,w)^{1/2}}{\Delta(z,w)^{2\sigma+1/2}}.$$

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  and this is exactly the issue that will save us!
- This places us in the setting of non-homogeneous harmonic analysis as developed by Nazarov, Treil and Volberg. We have an operator with a Calderón-Zygmund kernel satisfying estimates of order 2σ, a measure μ of order 2σ, and are interested in $L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$ bounds.

Euclidean Variant of the Question

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More precisely, for $m \leq d$ we are interested in Calderón-Zygmund kernels that satisfy the following estimates:

$$|k(x,y)| \leq \frac{C_{CZ}}{|x-y|^m},$$

and

$$|k(y,x)-k(y,x')|+|k(x,y)-k(x',y)| \leq C_{CZ}\frac{|x-x'|^\tau}{|x-y|^{m+\tau}}$$

provided that $|x-x'| \leq \frac{1}{2}|x-y|$, with some (fixed) $0<\tau\leq 1$ and $0<C_{CZ}<\infty$.

Euclidean Variant of the Question

Additionally the kernels will have the following property

$$|k(x,y)| \leq \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) := \mathrm{dist}(x, \mathbb{R}^d \setminus H)$ and H being an open set in $\mathbb{R}^d$.

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We will say that k is a Calderón-Zygmund kernel on a closed $X \subset \mathbb{R}^d$ if $k(x,y)$ is defined only on $X \times X$ and the previous properties of k are satisfied whenever $x, x', y \in X$.

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Once the kernel has been defined, then we say that a $L^2(\mathbb{R}^d; \mu)$ bounded operator is a Calderón-Zygmund operator with kernel k if,

$$T_{\mu,m} f(x) = \int_{\mathbb{R}^d} k(x,y) f(y) d\mu(y) \quad \forall x \notin \mathrm{supp} f.$$

Main Results

T(1)-Theorem for Bergman-Type Operators

Theorem (T(1)-Theorem for Bergman-Type Operators, Volberg and Wick, Amer. J. Math. to appear)

Let $k(x,y)$ be a Calderón-Zygmund kernel of order m on $X \subset \mathbb{R}^d$, $m \leq d$ with Calderón-Zygmund constants C_{CZ} and τ. Let μ be a probability measure with compact support in X and all balls such that $\mu(B_r(x)) > r^m$ lie in an open set H. Let also

$$|k(x,y)| \leq \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) := \operatorname{dist}(x, \mathbb{R}^d \setminus H)$. Finally, suppose also that:

$$\| T_{\mu,m}\chi_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \leq A\,\mu(Q)\,,\ \| T^*_{\mu,m}\chi_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \leq A\,\mu(Q)\,.$$

Then $\| T_{\mu,m}\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \leq C(A,m,d,\tau)$.

Remarks about T(1)-Theorem for Bergman-Type Operators

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  - This is just a restatement of the Carleson measure condition in this context.
- To handle this difficulty we suppose that our Calderón-Zygmund kernels have an additional estimate in terms of the behavior of the distance to the complement of H (namely that they are Bergman-type kernels).

Main Results

Characterization of Carleson Measures for $B_\sigma^2(\mathbb{B}_n)$

Theorem (Characterization of Carleson Measures for Besov-Sobolev Spaces, Volberg and Wick, Amer. J. Math. to appear)

Let μ be a positive Borel measure in $\mathbb{B}_n$. Then the following conditions are equivalent:

(a) *μ is a $B_2^\sigma(\mathbb{B}_n)$-Carleson measure;*

(b) *$T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$ is bounded;*

(c) *There is a constant C such that*

- (i) $\|T_{\mu,2\sigma}\chi_Q\|^2_{L^2(\mathbb{B}_n;\mu)} \leq C\,\mu(Q)$ *for all Δ-cubes Q;*
- (ii) $\mu(B_\Delta(x,r)) \leq C\, r^{2\sigma}$ *for all balls $B_\Delta(x,r)$ that intersect $\mathbb{C}^n \setminus \mathbb{B}_n$.*

Above, the sets B_Δ are balls measured with respect to the metric Δ and the set Q is a "cube" defined with respect to the metric Δ.

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- It only remains to prove that $(c) \Rightarrow (b)$.
  - The proof of this Theorem follows from the T(1)-Theorem for Bergman-type operators.
  - In a neighborhood of the sphere $\partial\mathbb{B}_n$ the metric Δ looks a Euclidean-type quasi-metric. For example when $n = 2$ we have that

$$\Delta(x, y) \approx |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|^2 + |x_4 - y_4|^2$$

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  - The method of proof of the Euclidean Bergman-type T(1) theorem can then be modified to case of Calderón-Zygmund operators with respect to a quasi-metric (essentially verbatim).
- It is possible to show that the T(1) condition reduces to the simpler conditions in certain cases.
- An alternate proof of this Theorem was recently given by Hytönen and Martikainen. Their proof used a non-homogeneous T(b)-Theorem on metric spaces.

Sketch of Proof

Littlewood-Paley Decomposition

- Construct two independent dyadic lattices $\mathcal{D}_1$ and $\mathcal{D}_2$.

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- There are special unit cubes Q^0 and R^0 of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively that transit cubes and contain suppμ deep inside them.
- Define expectation operators Δ_Q (Haar function on Q) and Λ (average on Q^0), then we have for every $\varphi \in L^2(\mathbb{R}^d;\mu)$

$$\varphi = \Lambda\varphi + \sum_{Q\in\mathcal{D}_1^{tr}} \Delta_Q\varphi,$$

  the series converges in $L^2(\mathbb{R}^d;\mu)$ and, moreover,

$$\|\varphi\|^2_{L^2(\mathbb{R}^d;\mu)} = \|\Lambda\varphi\|^2_{L^2(\mathbb{R}^d;\mu)} + \sum_{Q\in\mathcal{D}_1^{tr}} \|\Delta_Q\varphi\|^2_{L^2(\mathbb{R}^d;\mu)}.$$

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For a dyadic cube R we denote $\cup_{i=1}^{2^d} \partial R_i$ by *sk* R, called the *skeleton* of R. Here the R_i are the dyadic children of R.

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Definition (Bad Cubes)

Let τ, m be parameters of the Calderón-Zygmund kernel k. We fix $\alpha = \frac{\tau}{2\tau+2m}$. Fix a small number $\delta > 0$. Fix $S \geq 2$ to be chosen later. Choose an integer r such that

$$2^{-r} \leq \delta^S < 2^{-r+1}.$$

A cube $Q \in \mathcal{D}_1$ is called *bad* (or δ-bad) if there exists $R \in \mathcal{D}_2$ such that

- $\ell(R) \geq 2^r \ell(Q)$,
- $\operatorname{dist}(Q, sk\, R) < \ell(Q)^{\alpha} \ell(R)^{1-\alpha}$.

Sketch of Proof

Good and Bad Decomposition

We fix the decomposition of f and g into good and bad parts:

$$f = f_{good} + f_{bad}\,, \text{ where } f_{good} = \Lambda f + \sum_{Q \in \mathcal{D}_1^{tr} \cap \mathcal{G}_1} \Delta_Q f$$

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One can choose $S = S(\alpha)$ in such a way that for any fixed $Q \in \mathcal{D}_1$,

$$\mathbb{P}\{Q \text{ is bad}\} \leq \delta^2$$

$$\mathbb{E}(\|f_{bad}\|_{L^2(\mathbb{R}^d;\mu)}) \leq \delta \|f\|_{L^2(\mathbb{R}^d;\mu)}.$$

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Similar Statements for g hold as well.

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Reduction to Controlling The Good Part

- Using the decomposition above, we have

$$\langle T_{\mu,m} f, g\rangle_{L^2(\mathbb{R}^d;\mu)} = \langle T_{\mu,m} f_{good}, g_{good}\rangle_{L^2(\mathbb{R}^d;\mu)} + R(f,g)$$

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- Using the construction above, we have that

$$\mathbb{E}|R_\omega(f,g)| \leq 2\,\delta \|T\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \|f\|_{L^2(\mathbb{R}^d;\mu)} \|g\|_{L^2(\mathbb{R}^d;\mu)}.$$

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- Choosing δ small enough ($< \frac{1}{4}$) we only need to show that

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- This will then give

$$\|T\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \leq 2C(\tau, m, A, d)$$

Sketch of Proof

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- These all then imply that

$$\left| \langle T_{\mu,m} f_{good}, g_{good} \rangle_{L^2(\mathbb{R}^d;\mu)} \right| \leq C(\tau, m, A, d) \|f\|_{L^2(\mathbb{R}^d;\mu)} \|g\|_{L^2(\mathbb{R}^d;\mu)}.$$

Extension to "Nice" Metric Spaces

- It is possible to extend these results to Ahlfor regular metric spaces. Namely, (X, ρ, ν) with (X, ρ) a complete metric space and ν a Borel measure on X such that

$$\nu(B_r(x)) \approx r^n$$

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- Suppose we have another measure μ on the metric space X (which need not be doubling), but satisfies the following relationship, for some $0 \leq m < n$

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- A *standard Calderón-Zygmund kernel of order* $0 < m \leq n$ is a function $k : X \times X \setminus \{x = y\} \to \mathbb{C}$ such that there exists constants $C_{CZ}, \tau, \delta > 0$

$$|k(x,y)| \leq \frac{C_{CZ}}{\rho(x,y)^m} \quad \forall x \neq y \in X;$$

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and

$$|k(x,y) - k(x,y')| + |k(x,y) - k(x',y)| \leq C_{CZ} \frac{\rho(x,x')^{\tau}}{\rho(x,y)^{\tau+m}}$$

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- If we additionally suppose the kernels that have the additional property that

$$|k(x,y)| \leq \frac{1}{\max(d^m(x), d^m(y))},$$

where $d(x) := \mathrm{dist}(x, X \setminus \Omega) = \inf\{\rho(x,y) : y \in X \setminus \Omega\}$ and Ω being an open set in X.

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- Then the analogue of the Bergman-type T(1) Theorem holds with essentially the same proof.

Thank You!