Structure results on A_∞

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Harmonic Analysis: A Retrospective Workshop
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Maximal functions and A_∞: three approaches

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One damned question of R. Fefferman

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The question.

Somewhat over a decade ago, R. Fefferman asked me the following question: does the strong maximal operator M_s map rectangular A_∞ weights into rectangular A_1?

Background: Maximal Functions

- Recall the *Hardy-Littlewood maximal function* M, defined on $f \in L^1_{loc}(\mathbb{R})$ by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

- The *strong* maximal function M_s is defined on $f \in L^1_{loc}(\mathbb{R}^2)$ by

$$M_s f(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |f(y)|\, dy.$$

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The class A_p, $1 < p < \infty$

- Recall a non-negative function $w \in L^1_{loc}$ is an A_p *weight* $(p > 1)$ if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} < \infty.$$

- $w \in A_{p,s}$ $(p > 1)$ ("rectangular A_p") if

$$\sup_R \left(\frac{1}{|R|} \int_R w \right) \left(\frac{1}{|R|} \int_R w^{-\frac{1}{p-1}} \right)^{p-1} < \infty.$$

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Basics:

- $\cup_{p>1} A_p =: A_\infty$.
- There is a "best" A_p class $A_1 \subset \cap_p A_p$; viz., all w such that $Mw(x) \leq cw(x)$ a.e.
- $w \in A_p$ for some $p > 1$ if and only if $w \in RH_s$, for some $s > 1$, i.e., satisfies some *reverse Hölder inequality*,

$$\frac{1}{|Q|}\int_Q w \leq \left(\frac{1}{|Q|}\int_Q w^s\right)^{1/s} \leq C\frac{1}{|Q|}\int_Q w$$

for all cubes $Q \subset \mathbb{R}^n$.

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M maps A_∞ into A_1

First approach: Johnson-Neugebauer

- Coifman-Rochberg ('80): given any $f \in L^1_{loc}(\mathbb{R})$, any $\delta \in (0,1)$,
$$(Mf)^\delta \in A_1 .$$
- Since for any weight $w \in RH_s$,
$$Mw \approx (Mw^s)^{1/s},$$
we see Mw must be an A_1 weight.
- The trouble: for M_s, however, Coifman-Rochberg fails (counterexample, F. Soria).

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Trying to circumvent Coifman-Rochberg

Second approach: $M : A_\infty \to A_1$

- Closely related fact: $M : BMO \to BLO$, the functions of *bounded lower oscillation*, i.e., all ϕ such that $\sup_Q \frac{1}{|Q|} \int_Q \phi - inf_Q \phi < \infty$.
- (Recall $\log A_\infty \subset BMO$, $\log A_1 \subset BLO$)

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- Modify the maximal operator M: consider the *natural maximal operator* $M^\natural$ (introduced by C. Bennett), given by

$$M^\natural f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B f,$$

then

$$BLO = \{\phi \mid M^\natural \phi(x) \leq \phi(x) + C \text{ a.e.}\}.$$

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$M^\natural$ commutes with the logarithm (on A_∞):

- Any $w \in A_\infty$ satisfies a *reverse* Jensen inequality

$$e^{\frac{1}{|Q|}\int_Q \log w} \leq \frac{1}{|Q|}\int_Q w \leq A_\infty(w) e^{\frac{1}{|Q|}\int_Q \log w},$$

taking supremums and logarithms yields

$$0 \leq [\log M^\natural - M^\natural \log] w \leq \log A_\infty(w),$$

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We want to show $MMw(x) \leq CMw(x)$:

- Using the commutation twice, we get

$$e^{M^\natural M^\natural \log w} \leq MMw \leq A_\infty(w)A_\infty(Mw)e^{M^\natural M^\natural \log w}.$$

i.e.,

$$MMw \approx e^{M^\natural M^\natural \log w}$$

The boundedness of $M^\natural : BMO \to BLO$ implies

$$\leq e^{M^\natural \log w + ||M^\natural \log w||_{BLO}} \approx e^{||M^\natural \log w||_{BLO}} Mw$$

- So M maps A_∞ into A_1 again.
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Digression: Refined Jones Factorization

As a side bonus, commutation yields the following characterizations:

- $$w \in A_1 \iff w \in A_\infty \cap e^{BLO}$$

  $$w \in RH_\infty \iff w \in e^{BUO} (= e^{-BLO}).$$

- And these can be used to clarify a generalized Jones factorization due to Cruz-Uribe and Neugebauer:

  $$w \in A_p \cap RH_s \iff w = w_0 w_1,$$

  where $w_0 \in A_1 \cap RH_s$ and $w_1 \in A^p \cap RH_\infty$

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The crucial step in Cruz-Uribe–Neugebauer's proof of the generalized Jones factorization is the following.

- **Lemma:** $w \in A_1 \iff w^{1-p} \in A_p \cap RH_\infty$.

  Proof:

  $$w \in A_1 \iff w \in A_\infty \cap e^{BLO} \iff \left\{ \begin{array}{c} w \in A_{p'} \\ w^{1-p} \in e^{BUO} = RH_\infty \end{array} \right\}$$

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- **Proof** (of refined Jones factorization):
  $w \in A_p \cap RH_s \iff w^s \in A_{s(p-1)+1}$
  $\iff w^s = v_0 v_1^{1-[s(p-1)+1]} = v_0 v_1^{-s(p-1)}$ (by original Jones factorization)
  $\iff w = v_0^{1/s} v_1^{1-p}; v_0, v_1 \in A_1;$
  take $w_0 = v_0^{1/s}$, $w_1 = v_1^{1-p}$.
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Getting back to the question of $M_s : A_\infty \to A_1$

Third approach: the Bellman method.

- Bellman-type methods introduced by Burkholder in the '80s then used with great effect by Nazarov, Treil, and Volberg, starting in '95. Recently undergone great evolution, starting with Vasyunin in '03 (and then Dindos, Slavin, Stokolos, Vasyunin, Volberg, Wall, etc. to sharpen various results)
- E.g., Vasyunin ('03), Dindos and Wall ('06) on A_p and RH_s; Slavin, Stokolos, and Vasyunin ('08) used it to get sharp bounds for M on $L^p(\mathbb{R})$.

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The New Bellman Philosophy

Meta-observation: Many of the objects ("B", say) of interest depend on, or are relations between, "martingale variables," i.e., constructs V which satisfy a relation of the form

$$V = \frac{V_- + V_+}{2}.$$

Such objects/concepts themselves often also satisfy a "pseudo-concavity" condition

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These two facts together force the constructs of interest to be solutions of Monge-Ampère PDEs; further, such problems can be solved for *explicitly* (via "Bellman foliations (Vasyunin-Volberg)), yielding *sharp* results.

- *Hope:* to use the Bellman approach to re-prove the classical $M : A_\infty \to A_1$ (in the dyadic case) without Coifman-Rochberg. Then (if possible) generalize for M_s. (Work in progress with Slavin)

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Bellman Approach to $M : A_\infty \to A_1$

- Let M denote the *dyadic* maximal operator

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supremum of averages over all dyadic intervals $I \ni x$.

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($\Omega_\delta = \{(x_1, x_2, L) : \log(\frac{x_1}{\delta}) \le x_2 \le \log x_1; 0 < x_1 \le L\}$ is the domain of B).

- In the dyadic case, $L = \inf_I Mw$; so to show $Mw \in A_1$, "all" we must do is show that $B(x_1, x_2, L) \le CL$.

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Problem: how do we figure out what B is?

Simple observations about B

- *Homogeneity:* If instead we take the supremum over $\tilde{w} = \tau w \in A_\infty^\delta$, $\tau > 0$, we see that

$$\tau B(x_1, x_2, L) = B(\tau x_1, x_2 + \log \tau, \tau L).$$

Differentiating with respect to τ and setting $\tau = 1$ yields

$$x_1 B_{x_1} + B_{x_2} + L B_L = B$$

- *Boundary condition:* On the boundary $x_2 = \log x_1$ of Ω_δ, w must be constant; thus

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Key observation about B: pseudoconcavity

- *Pseudoconcavity:* Fix $L > 0$, and choose x^-, x^+ such that $\frac{1}{2}(x_1^- + x_1^+) \leq L$. Taking the supremum of

$$\langle Mw \rangle_I = \frac{1}{2}\langle Mw \rangle_{I^-} + \frac{1}{2}\langle Mw \rangle_{I^+}$$

over weights $w \in A_\infty^\delta$ such that $(\langle w \rangle_{I^\pm}, \langle \log w \rangle_{I^\pm}) = x^\pm$, we see that (since $\sup_{R\supset Q_\pm}\langle w \rangle_R = \max\left\{\sup_{R\supset Q}\langle w \rangle_R, x_1^\pm\right\}$),

$$B(\frac{1}{2}(x^- + x^+), L) \geq \frac{1}{2}B(x^-, \max\{L, x_1^-\}) + \frac{1}{2}B(x^+, \max\{L, x_1^+\}).$$

- The above is the key to getting that differential inequalities that lead to a Monge-Ampère problem.

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Getting differential inequalities

Notice that if, in the pseudoconcavity condition, we take $x^-, x^+ \in \omega_\delta$ close enough to x, and such that $x = \frac{1}{2}(x^- + x^+)$. then $L^\pm = \max\{L, x_1^\pm\} = L$ and so pseudoconcavity becomes concavity:

$$B(x, L) \geq \frac{1}{2}B(x^-, L) + \frac{1}{2}B(x^+, L).$$

This means that, everywhere in Ω_δ where B is sufficiently differentiable, we must have negative semidefiniteness of the Hessian of B,

$$\mathrm{d}_x^2 B = \begin{bmatrix} B_{x_1 x_1} & B_{x_1 x_2} \\ B_{x_1 x_2} & B_{x_2 x_2} \end{bmatrix}.$$

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Getting differential inequalities: Sketch

On the infinitesimal scale, the pseudoconcavity condition (plus the fact that x_1 and x_2 are "martingale variables" forces (on the boundary $L = x_1$),

$$-\frac{1}{8}(\Delta x)^T \mathrm{d}_x^2 B(x, x_1)(\Delta x) + \frac{1}{2} B_L(x, x_1)(x_1 - x_1^+) + o(\|\Delta x\|^2) \geq 0.$$

The first term is non-negative and of the second order, while the second term is non-positive (since $B_L \geq 0$ and (wlog) $x_1 < x_1^+$) and of the first order. Thus we must have

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The Monge-Ampère Boundary Value Problem

- Further, roughly speaking, the fact that the Bellman function is the minimum of all candidates satisfying the pseudoconcavity condition causes us to demand that the Hessian be singular, i.e.,

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- All together, we have the following Monge-Ampère boundary value problem:

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Theorem (Vasyunin-Volberg)

Let Ω be a plane domain and $G = G(x_1, x_2)$ be a C^2 function satisfying the homogeneous Monge–Ampère equation in Ω :

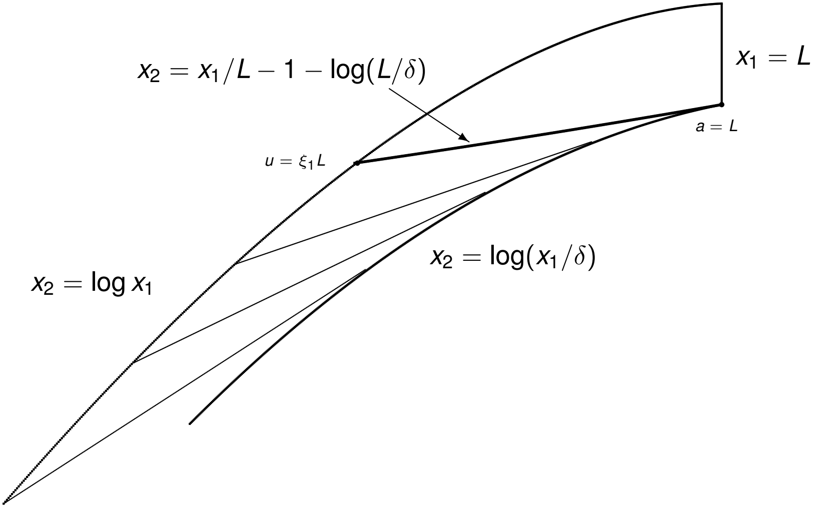
$$G_{x_1 x_1} G_{x_2 x_2} = G^2_{x_1 x_2}, \tag{1}$$

and such that either $G_{x_1 x_1} \neq 0$ or $G_{x_2 x_2} \neq 0$.
Then there are differentiable functions t_0, t_1, t_2 of (x_1, x_2) such that

$$G = t_1 x_1 + t_2 x_2 + t_0 \tag{2}$$

and t_k are constant on each integral trajectory generated by the kernel of the Hessian $\mathrm{d}^2 G$. Moreover, these integral trajectories are straight lines given by

$$x_1 dt_1 + x_2 dt_2 + dt_0 = 0. \tag{3}$$


$x_2 = x_1/L - 1 - \log(L/\delta)$
$x_1 = L$
$a = L$
$u = \xi_1 L$
$x_2 = \log(x_1/\delta)$
$x_2 = \log x_1$

- Solving for the Bellman candidate in the lower region, it turns out to be

$$B(x) = \frac{1}{\xi}\left(\frac{a}{L}\right)^{\xi/(1-\xi)}(x_1 - a\xi) + L,$$

where a is the x_1 coordinate of the tangent point $(a, \log(a/\delta))$ of the line of foliation

$$x_2 = \frac{x_1}{a} - 1 + \log\left(\frac{a}{\delta}\right),$$

and ξ the lesser root of the equation $\xi - \log\xi = 1 + \log\delta$.

- We also have an expression for the Bellman candidate in the upper region, and the bound C appears to be $\frac{1}{\xi}$. (Still not finished: need to check various points.)
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