Corona Theorems for Multiplier Algebras on $\mathbb{B}_n$

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The Corona Problem for $H^\infty(\mathbb{D})$

- The Banach algebra $H^\infty(\mathbb{D})$ is the collection of all analytic functions on the disk such that

$$||f||_{H^\infty(\mathbb{D})} = \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

- To each $z \in \mathbb{D}$ we can associate a multiplicative linear functional on $H^\infty(\mathbb{D})$ (point evaluation at z)

$$\varphi_z(f) = f(z)$$

- Let Δ denote the maximal ideal space of $H^\infty(\mathbb{D})$. (Maximal ideals = kernels of nontrivial multiplicative linear functionals.)

Gelfand Theory of Commutative Banach Algebras

- Suppose X is a commutative Banach algebra with an identity element and let Δ be its maximal ideal space. (A maximal ideal is the kernel of a nontrivial multiplicative linear functional on X.) When equipped with the weak topology, Δ is a compact Hausdorff space. Define the Gelfand transform $\Lambda : X \to C(\Delta)$, $\hat{x}(h) = h(x)$.
- Then the Gelfand transform is a continuous algebra homomorphism from X onto a subalgebra $\widehat{X}$ of $C(\Delta)$ with kernel the intersection of all maximal ideals.
- One would like to know more about the compact Hausdorff space Δ in specific situations. For example, when X is a nice Banach algebra of functions on a set Ω, then Ω is embedded in Δ via the point evaluations, and the question arises as to whether or not Ω is dense in Δ.

The Corona Problem for $H^\infty(\mathbb{D})$

In 1941, Kakutani asked if there was a Corona in the maximal ideal space Δ of $H^\infty(\mathbb{D})$, i.e. whether or not the disk $\mathbb{D}$ was dense in Δ. One then defines the Corona of the algebra to be $\Delta \setminus \overline{\mathbb{D}}$.

In 1962, Lennart Carleson proved the absence of a Corona by showing that if $\{g_j\}_{j=1}^N$ is a finite set of functions in $H^\infty(\mathbb{D})$ satisfying

$$0 < \delta \leq \sum_{j=1}^{N} |g_j(z)|^2 \leq 1, \qquad z \in \mathbb{D},$$

then there are functions $\{f_j\}_{j=1}^N$ in $H^\infty(\mathbb{D})$ with

$$\sum_{j=1}^{N} f_j(z)\, g_j(z) = 1, \quad z \in \mathbb{D} \text{ and } \sum_{j=1}^{N} ||f_j||_{H^\infty(\mathbb{D})} \leq C(\delta).$$

Extensions of the Corona Problem

The point of departure for many generalizations of Carleson's Corona Theorem is the following:

Observation

$H^\infty(\mathbb{D})$ is the (pointwise) multiplier algebra of the classical Hardy space $H^2(\mathbb{D})$ on the unit disc.

Namely, let $M_{H^2(\mathbb{D})}$ denote the class of functions φ such that

$$||M_\varphi f||_{H^2(\mathbb{D})} \leq C||f||_{H^2(\mathbb{D})}, \quad \forall f \in H^2(\mathbb{D}) \tag{1}$$

with $||\varphi||_{M_{H^2(\mathbb{D})}} = \inf\{C : (1) \text{ holds}\}$. Here M_φ is the multiplier operator on $H^2(\mathbb{D})$ defined by $M_\varphi f = \varphi f$. It can be shown that $\varphi \in H^\infty(\mathbb{D})$ if and only if $\varphi \in M_{H^2(\mathbb{D})}$ and $||\varphi||_{M_{H^2(\mathbb{D})}} = ||\varphi||_{H^\infty(\mathbb{D})}$.

Besov-Sobolev Spaces

- The spaces $B_2^\sigma(\mathbb{B}_n)$ are examples of reproducing kernel Hilbert spaces.
- Namely, for each point $\lambda \in \mathbb{B}_n$ there exists a function $k_\lambda \in B_2^\sigma(\mathbb{B}_n)$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle_{B_2^\sigma}.$$

- It isn't too difficult to show that the kernel function k_λ is given by

$$k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^{2\sigma}}.$$

  - $\sigma = \frac{1}{2}$: the Drury-Arveson Hardy space H_n^2; $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)}$.
  - $\sigma = \frac{n}{2}$: the classical Hardy space $H^2(\mathbb{B}_n)$; $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^n}$.
  - $\sigma = \frac{n+1}{2}$: the Bergman space $\mathcal{B}(\mathbb{B}_n)$; $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$.

Besov-Sobolev Spaces

- The space $B_2^\sigma(\mathbb{B}_n)$ consists of all holomorphic functions f on the unit ball $\mathbb{B}_n$ such that

$$\left(\sum_{k=0}^{m-1}\left|f^{(k)}(0)\right|^2+\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma}f^{(m)}(z)\right|^2 d\lambda_n(z)\right)^{\frac{1}{2}}<\infty,$$

  where $d\lambda_n(z)=\left(1-|z|^2\right)^{-n-1}dV(z)$ is the invariant measure on $\mathbb{B}_n$, $\left(1-|z|^2\right)f'(z)$ is an "invariant" derivative and $m>\frac{n}{2}-\sigma$.
  - $\sigma<0$: $B_2^\sigma(\mathbb{B}_n)$ is contained in $C(\overline{\mathbb{B}_n})$ and $\Delta=\overline{\mathbb{B}_n}$.
  - $\sigma=0$: the Dirichlet space $\mathcal{D}(\mathbb{B}_n)$;
  - $\sigma=\frac{1}{2}$: the Drury-Arveson Hardy space H_n^2;
  - $\sigma=\frac{n}{2}$: the Classical Hardy space $H^2(\mathbb{B}_n)$;
  - $\sigma=\frac{n+1}{2}$: the Bergman space $\mathcal{B}(\mathbb{B}_n)$.

The DA Hardy space

The first hint that the classical Hardy space $H^2(\mathbb{B}_n)$ (consisting of holomorphic functions on the ball with $L^2(\sigma)$ boundary values) may not be the correct generalization of the classical Hardy space on the disk came with the failure of the von Neumann's inequality in higher dimensions. Recall the classical inequality:

Theorem (von Neumann 1951 [19])

Let H be a Hilbert space and let f be a complex-valued polynomial. Then for any contraction T on H,

$$||f(T)||_{H\to H} \leq ||f(S^*)||_{H^2\to H^2} = ||f||_{H^\infty(\mathbb{D})},$$

where S^* is the backward shift operator on $H^2 = H^2(\mathbb{D})$.

Drury found the correct generalization to the multivariable setting.

Drury's generalization

- Let $A = (A_1, \ldots, A_n)$ be an n-contraction on a complex Hilbert space $\mathcal{H}$ with pairwise commuting components satisfying $\sum_{j=1}^n ||A_j h||^2 \leq ||h||^2$ for all $h \in \mathcal{H}$
- Drury showed in 1978 [15] that if f is a complex polynomial on $\mathbb{C}^n$, then $||f(A)|| \leq ||f||_{M_{\mathcal{K}(\mathbb{B}_n)}}$ for all n-contractions A on $\mathcal{H}$ where $||f(A)||$ is the operator norm of $f(A)$ on $\mathcal{H}$, and $||f||_{M_{\mathcal{K}(\mathbb{B}_n)}}$ denotes the multiplier norm of the polynomial f on Drury's Hardy space of holomorphic functions

$$\mathcal{K}(\mathbb{B}_n) = \left\{ \sum_k a_k z^k, z \in \mathbb{B}_n : \sum_k |a_k|^2 \frac{k!}{|k|!} < \infty \right\},$$

  denoted H_n^2 by Arveson in 1998 [8], who also proves Drury's inequality.
- Moreover, equality holds above when $A = (S_1^*, \ldots, S_n^*)$.

Chen's identification of the DA space

- In 2003 Chen [12] has identified the Drury-Arveson Hardy space $\mathcal{K}(\mathbb{B}_n) = H_n^2$ as the Besov-Sobolev space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$ consisting of those holomorphic functions $\sum_k a_k z^k$ in the ball with coefficients a_k satisfying

$$\sum_k |a_k|^2 \frac{|k|^{n-1}(n-1)!k!}{(n-1+|k|)!} < \infty.$$

- The multiplier norms are equivalent: $||f||_{M_{\mathcal{K}(\mathbb{B}_n)}} \approx ||f||_{M_{B_2^{\frac{1}{2}}(\mathbb{B}_n)}}$.
- We note that a number of operator-theoretic properties of the space H_n^2 are developed by Arveson in [8], including model theory, that establish its central position in multivariable operator theory.

Multiplier Algebras of Besov-Sobolev Spaces $M_{B_2^\sigma(\mathbb{B}_n)}$

- We are interested in the multiplier algebras $M_{B_2^\sigma(\mathbb{B}_n)}$ for $B_2^\sigma(\mathbb{B}_n)$. A function φ belongs to $M_{B_2^\sigma(\mathbb{B}_n)}$ if

$$\begin{aligned} \|\varphi f\|_{B_2^\sigma(\mathbb{B}_n)} &\leq C\|f\|_{B_2^\sigma(\mathbb{B}_n)}, \quad \forall f \in B_2^\sigma(\mathbb{B}_n) \\ \|\varphi\|_{M_{B_2^\sigma(\mathbb{B}_n)}} &= \inf\{C : \text{ the above inequality holds}\}. \end{aligned}$$

- It can be shown that $M_{B_2^\sigma(\mathbb{B}_n)} = H^\infty(\mathbb{B}_n) \cap \mathcal{X}_2^\sigma(\mathbb{B}_n)$, where $\mathcal{X}_2^\sigma(\mathbb{B}_n)$ is the collection of functions φ such that $d\mu_{\varphi,m} = \left|\left(1-|z|^2\right)^{m+\sigma}\varphi^{(m)}(z)\right|^2 d\lambda_n(z)$ is a Carleson measure. That is, for all $f \in B_2^\sigma(\mathbb{B}_n)$:

$$\int_{\mathbb{B}_n} |f(z)|^2 \left|\left(1-|z|^2\right)^{m+\sigma}\varphi^{(m)}(z)\right|^2 d\lambda_n(z) \leq C^2\|f\|^2_{B_2^\sigma(\mathbb{B}_n)}$$

with $\|\varphi\|_{\mathcal{X}_2^\sigma(\mathbb{B}_n)} = \inf\{C : \text{ the above inequality holds}\}$.

Multiplier Algebras of Besov-Sobolev Spaces $M_{B_2^\sigma(\mathbb{B}_n)}$

- The boundedness of multipliers is shown via a standard argument. Namely, if $\varphi \in M_{B_2^\sigma(\mathbb{B}_n)}$, then $\varphi = M_\varphi 1$ is in $B_2^\sigma = B_2^\sigma(\mathbb{B}_n)$ since $1 \in B_2^\sigma$. Here M_φ is the multiplier operator on B_2^σ defined by $M_\varphi f = \varphi f$. The adjoint M_φ^* of the multiplier operator M_φ is bounded on $(B_2^\sigma)'$, and if e_z is the point evaluation functional at z on B_2^σ, then
$$\langle f, M_\varphi^* e_z\rangle = \langle M_\varphi f, e_z\rangle = \varphi(z)f(z) = \varphi(z)\langle f, e_z\rangle = \langle f, \overline{\varphi(z)}e_z\rangle$$
for every $f \in B_2^\sigma(\mathbb{B}_n)$. This shows that $M_\varphi^* e_z = \overline{\varphi(z)}e_z$. Thus
$$|\varphi(z)|\,||e_z||_{(B_2^\sigma)'} = ||M_\varphi^* e_z||_{(B_2^\sigma)'} \leq ||M_\varphi^*||\,||e_z||_{(B_2^\sigma)'}.$$
This implies that $|\varphi(z)| \leq ||M_\varphi^*|| = ||M_\varphi||$ since $||e_z||_{(B_2^\sigma)'} < \infty$.
- We have $||\varphi||_{M_{B_2^\sigma(\mathbb{B}_n)}} \approx ||\varphi||_{H^\infty(\mathbb{B}_n)} + ||\varphi||_{\chi_2^\sigma(\mathbb{B}_n)}$.

The Corona Problem for $M_{B_2^\sigma(\mathbb{B}_n)}$

We wish to study a generalization of Carleson's Corona Theorem to higher dimensions and additional function spaces.

Question (Corona Problem)

Given $g_1, \ldots, g_N \in M_{B_2^\sigma(\mathbb{B}_n)}$ *satisfying* $0 < \delta \leq \sum_{j=1}^N |g_j(z)|^2 \leq 1$ *for all* $z \in \mathbb{B}_n$, *does there exist a constant* $C_{n,\sigma,N,\delta}(g)$ *and functions* $f_1, \ldots, f_N \in M_{B_2^\sigma(\mathbb{B}_n)}$ *satisfying*

$$\begin{aligned} \sum_{j=1}^N \|f_j\|_{M_{B_2^\sigma(\mathbb{B}_n)}} &\leq C_{n,\sigma,N,\delta}(g) \\ \sum_{j=1}^N f_j(z) g_j(z) &= 1, \quad z \in \mathbb{B}_n? \end{aligned}$$

The Baby Corona Problem

It is easy to see that the Corona Problem for $M_{B_2^\sigma(\mathbb{B}_n)}$ implies a "simpler" question that one can consider.

Question (Baby Corona Problem)

Suppose we are given $g_1, \ldots, g_N \in M_{B_2^\sigma(\mathbb{B}_n)}$ *satisfying* $0 < \delta \leq \sum_{j=1}^N |g_j(z)|^2 \leq 1$ *for all* $z \in \mathbb{B}_n$. *Does there exist a constant* $C_{n,\sigma,N,\delta}(g)$ *such that for every* $h \in B_2^\sigma(\mathbb{B}_n)$ *there exist functions* $f_1, \ldots, f_N \in B_2^\sigma(\mathbb{B}_n)$ *satisfying*

$$\begin{aligned} \sum_{j=1}^N \|f_j\|^2_{B_2^\sigma(\mathbb{B}_n)} &\leq C_{n,\sigma,N,\delta}(g)\|h\|^2_{B_2^\sigma(\mathbb{B}_n)} \\ \sum_{j=1}^N f_j(z) g_j(z) &= h(z), \quad z \in \mathbb{B}_n? \end{aligned}$$

Baby Corona Theorem for $B_p^\sigma(\mathbb{B}_n)$

Theorem (ȘC, Eric T. Sawyer, Brett D. Wick (2008))

Let $0 \leq \sigma$ and $1 < p < \infty$. Given $g_1, \ldots, g_N \in M_{B_p^\sigma(\mathbb{B}_n)}$ satisfying $0 < \delta \leq \sum_{j=1}^N |g_j(z)|^2 \leq 1, \quad z \in \mathbb{B}_n$, there exists a constant $C_{n,\sigma,N,p,\delta}(g)$ such that for each $h \in B_p^\sigma(\mathbb{B}_n)$ there are functions $f_1, \ldots, f_N \in B_p^\sigma(\mathbb{B}_n)$ satisfying

$$\begin{aligned} \sum_{j=1}^N ||f_j||^p_{B_p^\sigma(\mathbb{B}_n)} &\leq C_{n,\sigma,N,p,\delta}(g)||h||^p_{B_p^\sigma(\mathbb{B}_n)} \\ \sum_{j=1}^N f_j(z)g_j(z) &= h(z), \quad z \in \mathbb{B}_n. \end{aligned}$$

The Corona Theorem for $M_{B_2^\sigma(\mathbb{B}_n)}$

Corollary (ŞC, Eric T. Sawyer, Brett D. Wick (2008))

Let $0 \leq \sigma \leq \frac{1}{2}$. *Given* $g_1, \ldots, g_N \in M_{B_2^\sigma(\mathbb{B}_n)}$ *satisfying* $0 < \delta \leq \sum_{j=1}^N |g_j(z)|^2 \leq 1, \quad z \in \mathbb{B}_n$, *there exists a constant* $C_{n,\sigma,N,\delta}(g)$ *and there exist functions* $f_1, \ldots, f_N \in M_{B_2^\sigma(\mathbb{B}_n)}$ *satisfying*

$$\begin{aligned} \sum_{j=1}^N \|f_j\|_{M_{B_2^\sigma(\mathbb{B}_n)}} &\leq C_{n,\sigma,N,\delta}(g) \\ \sum_{j=1}^N f_j(z) g_j(z) &= 1, \quad z \in \mathbb{B}_n. \end{aligned}$$

The Corona Theorem for $M_{B_2^\sigma(\mathbb{B}_n)}$

The proof of this Corollary follows easily from the main Theorem.

- When $0 \leq \sigma \leq \frac{1}{2}$, the spaces $B_2^\sigma(\mathbb{B}_n)$ are reproducing kernel Hilbert spaces with a complete Nevanlinna-Pick kernel.
- By the Toeplitz Corona Theorem, we have then that the Baby Corona Problem is equivalent to the full Corona Problem. The result then follows.

An additional corollary of the above result is the following:

Corollary

For $0 \leq \sigma \leq \frac{1}{2}$, the unit ball $\mathbb{B}_n$ is dense in the maximal ideal space of $M_{B_2^\sigma(\mathbb{B}_n)}$.

This is because the density of the unit ball $\mathbb{B}_n$ in the maximal ideal space of $M_{B_2^\sigma(\mathbb{B}_n)}$ is equivalent to the Corona Theorem above.

Earlier work

- In 2000 J. Ortega and J. Fabrega [20] obtained partial results with $N = 2$ generators for the Banach spaces $B_p^\sigma(\mathbb{B}_n)$ with $\sigma \in \left[0, \frac{1}{p}\right) \cup \left(\frac{n}{p}, \infty\right)$ and $1 < p < \infty$; and also for the case $N = 2$ with $\sigma = \frac{n}{p}$ when $1 < p \leq 2$.
- In 2000 M.Andersson and H. Carlsson [4] solved the baby corona problem for $H^2(\mathbb{B}_n)$ with $N = \infty$ and obtained the analogous (baby) H^p corona theorem on the ball B_n for $1 < p < \infty$ (see also Amar [2], Andersson-Carlsson [5], [3], and Krantz-Li [17].)
- In 2005 S. Treil and the third author [26] obtained the H^p corona theorem for the polydisk $\mathbb{D}^n$ (see also Lin [18] and Trent [27].)

Sketch of Proof of the Baby Corona Theorem

Suppose we are given $g_1, \ldots, g_N \in M_{B_p^\sigma(\mathbb{B}_n)}$ satisfying
$0 < \delta \leq \sum_{j=1}^N |g_j(z)|^2 \leq 1, \quad z \in \mathbb{B}_n.$

- Set $\varphi_j(z) = \frac{\overline{g_j(z)}}{\sum_i |g_i(z)|^2} h(z)$.
  - This solution is smooth and satisfies the correct estimates, but is far from analytic.
  - In order to have an analytic solution, we will need to solve a sequence of $\overline{\partial}$-equations:
    - For η a $\overline{\partial}$-closed $(0, q)$ form, we want to solve the equation $\overline{\partial}\psi = \eta$ for a $(0, q-1)$ form ψ.
    - To accomplish this, we will use the Koszul complex. This gives an algorithmic way of solving the $\overline{\partial}$-equations for each $(0, q)$ form with $1 \leq q \leq n$ after starting with a $(0, n)$ form.
    - This produces a correction to the initial guess of φ_j, call it ξ_j, and set $f_j = \varphi_j - \xi_j$. By the Koszul complex we will have that each f_j is in fact analytic. We also have $\sum f_j g_j = h$ but now the estimates that we seek are in doubt.

Sketch of Proof of the Baby Corona Theorem

- To guarantee the estimates, we have to look closer at the solution operator to the $\overline{\partial}$-equation on $\overline{\partial}$-closed $(0,q)$ forms. Following the work of Øvrelid and Charpentier, one can compute that the solution operator is an integral operator that takes $(0,q)$ forms to $(0,q-1)$ forms with integral kernel:

$$\frac{(1-w\overline{z})^{n-q}\left(1-|w|^2\right)^{q-1}}{\triangle(w,z)^n}(\overline{w}_j-\overline{z}_j),\qquad \forall 1\leq q\leq n.$$

Here $\triangle(w,z)=|1-w\overline{z}|^2-\left(1-|w|^2\right)\left(1-|z|^2\right)$. When $n=1$ the Charpentier kernel is the Cauchy kernel.

Sketch of Proof of the Baby Corona Theorem

- One then needs to show that these solution operators map the Besov-Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$ to themselves. This is accomplished by a couple of key facts:
  - The Besov-Sobolev spaces are very "flexible" in terms of the norm that one can use. We only need to take the parameter m sufficiently high.
  - We show that these operators are very well behaved on "real-variable" versions of the space $B_p^\sigma(\mathbb{B}_n)$. These, of course, contain the spaces that we are interested in.
  - To show that the solution operators are bounded on $L^p(\mathbb{B}_n; dV)$ the original proof uses the Schur test. To handle the boundedness on $B_p^\sigma(B_n)$, we can also use the Schur test but this requires a little more work to handle the derivative.
  - Finally, key to this approach, properties of the kernel and the unit ball are exploited to achieve the desired estimates.

The $H^\infty(\mathbb{B}_n)$ Corona Problem

When $\sigma = \frac{n}{2}$ (the classical Hardy space $H^2(\mathbb{B}_n)$), we have a weaker version of the Corona Problem that we can prove.

Theorem (ŞC, Eric T. Sawyer, Brett D. Wick (2009))

Given $g_1, \ldots, g_N \in H^\infty(\mathbb{B}_n)$ satisfying $0 < \delta \leq \sum_{j=1}^N |g_j(z)|^2 \leq 1$ for all $z \in \mathbb{B}_n$, there is a constant $C_{n,N,\delta}(g)$ and there are functions $f_1, \ldots, f_N \in BMOA(\mathbb{B}_n)$ satisfying

$$\sum_{j=1}^N ||f_j||_{BMOA(\mathbb{B}_n)} \leq C_{n,N,\delta}(g) \text{ and } \sum_{j=1}^N f_j(z)g_j(z) = 1, \quad z \in \mathbb{B}_n.$$

This gives another proof of a famous theorem of Varopoulos.

Open Problems and Future Directions

We know that the Corona Problem always implies the Baby Corona Problem. By the Toeplitz Corona Problem, we know that, under certain conditions on the reproducing kernel, these problems are in fact equivalent. What happens if we don't have these conditions ?

- Does the algebra $H^\infty(\mathbb{B}_n)$ of bounded analytic functions on the ball have a Corona in its maximal ideal space ?
- Does the Corona Theorem for the multiplier algebra of the Drury-Arveson space extend to more general domains in $\mathbb{C}^n$?
- Can we prove a Corona Theorem for *any* algebra in higher dimensions that is not the multiplier algebra of a Hilbert space with the complete Nevanlinna-Pick property ?

THANK YOU !!!

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