The Bellman Function for a Perturbation of Burkholder's Martingale Transform

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Motivation

Let $\{f_k\}_k$ be a martingale and $d_k = f_k - f_{k-1}, d_0 = f_0$ its associated martingale difference sequence. We define a martingale transform, $MT_{\vec{\varepsilon}}$, as

$$MT_{\vec{\varepsilon}}\left(\sum_{k=1}^{n} d_k\right) := \sum_{k=1}^{n} \varepsilon_k d_k,$$

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Theorem

(D. Burkholder–1983) For $1 < p < \infty$,
$\|MT_{\vec{\varepsilon}}\|_{p\to p} = \sup_{\vec{\varepsilon}} \frac{\left\|\sum_{k=1}^{n} \varepsilon_k d_k\right\|_p}{\|\sum_{k=1}^{n} d_k\|_p} = (p^* - 1)$, *where*
$p^* - 1 = \max\{p - 1, \frac{1}{p-1}\}$.

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This result has many applications, but the one we will focus on is sharp estimates of singular integrals.

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Geiss, Montgomery-Smith, Saksman (2008):
$\|\Re B\|_p = \|\Im B\|_p = p^* - 1$

Main Result

Goal: For $|\tau| \le \frac{1}{2}$, compute

$$C_{p,\tau} := \left\| \begin{pmatrix} MT_{\vec{\varepsilon}} \\ \tau I \end{pmatrix} \right\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R}^2)} = \sup_{\vec{\varepsilon}} \frac{\left\| \sum_{k=1}^n \begin{pmatrix} \varepsilon_k \\ \tau \end{pmatrix} d_k \right\|_p}{\| \sum_{k=1}^n d_k \|_p},$$

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This requires us to find the best $C_{p,\tau}$ such that for all $n \in \mathbb{Z}_+$

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Approach: Use the Bellman function technique

Let I be a finte interval and $\alpha^{\pm} \in \mathbb{R}^{+}$ such that $\alpha^{+} + \alpha^{-} = 1$.

Background Information and Notation

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These $\alpha^{\pm}$ generate two subintervals $I^{\pm}$ such that $|I^{\pm}| = \alpha^{\pm}|I|$ and $I = I^{-} \cup I^{+}$.

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Continuing this, for

$$\{\alpha_{n,m} : 0 < \alpha_{n,m} < 1, 0 \leq m < 2^n, 0 < n < \infty, \alpha_{n,2k} + \alpha_{n,2k+1} = 1\}$$

we generate $\mathcal{I} := \{I_{n,m} : 0 \leq m < 2^n, 0 < n < \infty\}$

where $I_{n,m} = I^-_{n,m} \cup I^+_{n,m} = I_{n+1,2m+1} \cup I_{n+1,2m+1}$ and

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Definition

For $J \in \mathcal{I}$ we define the Haar function
$h_J := -\sqrt{\frac{\alpha^+}{\alpha^-|J|}}\chi_{J^-} + \sqrt{\frac{\alpha^-}{\alpha^+|J|}}\chi_{J^+}$.

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Then for $1 < p < \infty$ and $f \in L^p[0,1)$,

$$f = \langle f \rangle_{[0,1)} \chi_{[0,1)} + \sum_{I \in \sigma(\mathcal{D})} (f, h_I) h_I.$$

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Definition

We define the martingale transform, g of f, in terms of the expansion in the Haar system, as

$$g := \langle f \rangle_{[0,1)} \chi_{[0,1)} + \sum_{I \in \sigma(\mathcal{D})} \varepsilon_I (f, h_I) h_I,$$

where $\varepsilon_I \in \{\pm 1\}$.

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$$\left\| \begin{pmatrix} \sum_{k=1}^n \varepsilon_k d_k \\ \tau \sum_{k=1}^n d_k \end{pmatrix} \right\|_{L^p([0,1),\mathbb{R}^2)} \leq C_{p,\tau} \left\| \sum_{k=1}^n d_k \right\|_{L^p([0,1),\mathbb{R})}$$

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So we define the Bellman function as

$$\mathcal{B}(x_1,x_2,x_3) := \sup_{f,g}\{\langle (g^2+\tau^2 f^2)^{\frac{p}{2}}\rangle_I : x_1 = \langle f\rangle_I, x_2 = \langle g\rangle_I, \\ x_3 = \langle |f|^p\rangle_I, |(f,h_J)| = |(g,h_J)|, \ \forall J \in \mathcal{D}\}$$

on the domain $\Omega = \{x \in \mathbb{R}^3 : x_3 \geq 0, |x_1|^p \leq x_3\}$.

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Computation of Bellman function when $p = 2$

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Now we can compute $\mathcal{B}$ explicitly.

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\end{aligned}$$

QED

Finding the Bellman function when $p \neq 2$ is much more difficult so we need some properties.

Weak concavity of Bellman function

Proposition

Suppose $x^{\pm} \in \Omega$ *such that* $x = \alpha^+ x^+ + \alpha^- x^-$, $\alpha^+ + \alpha^- = 1$. *If* $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ *then* $\mathcal{B}(x) \geq \alpha^+ \mathcal{B}(x^+) + \alpha^- \mathcal{B}(x^-)$

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$x^{\pm} \in \Omega$ s.t. $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$, iff $y^{\pm} \in \Xi$ satisfies either y_1 is fixed as $y_1^+ = y_1^-$ or y_2 is fixed as $y_2^+ = y_2^-$.

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Suppose $x^{\pm} \in \Omega$ such that $x = \alpha^+ x^+ + \alpha^- x^-$, $\alpha^+ + \alpha^- = 1$. If $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ then $\mathcal{B}(x) \geq \alpha^+ \mathcal{B}(x^+) + \alpha^- \mathcal{B}(x^-)$

Changing variables: $y_1 := \frac{x_2 + x_1}{2}$, $y_2 := \frac{x_2 - x_1}{2}$ and $y_3 := x_3$

So $\mathcal{M}(y_1, y_2, y_3) := \mathcal{B}(x_1, x_2, x_3) = \mathcal{B}(y_1 - y_2, y_1 + y_2, y_3)$ with the domain of $\mathcal{M}$ as $\Xi := \{y \in \mathbb{R}^3 : y_3 \geq 0, |y_1 - y_2|^p \leq y_3\}$.

$x^{\pm} \in \Omega$ s.t. $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$, iff $y^{\pm} \in \Xi$ satisfies either y_1 is fixed as $y_1^+ = y_1^-$ or y_2 is fixed as $y_2^+ = y_2^-$.

If $j \neq i \in \{1, 2\}$ and we fix y_i as $y_i^+ = y_i^-$. Then $\mathcal{M}$ as a function of y_j, y_3 is concave, i.e.

$$\begin{pmatrix} \mathcal{M}_{y_j y_j} & \mathcal{M}_{y_j y_3} \\ \mathcal{M}_{y_3 y_j} & \mathcal{M}_{y_3 y_3} \end{pmatrix} \leq 0,$$

which is equivalent to

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The Bellman function many other nice properties.

Proposition

Suppose $\mathcal{M}$ is $C^1(\mathbb{R}^3)$, then $\mathcal{M}$ has the following properties.

(i) Symmetry: $\mathcal{M}(y_1, y_2, y_3) = \mathcal{M}(y_2, y_1, y_3) = \mathcal{M}(-y_1, -y_2, y_3)$

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This is all of the properties of the Bellman function. Before we can begin to find an explicit formula we need to address a difficulty.

A needed assumption

Recall that one of the conditions for weak concavity of the Bellman function is $D_j = \mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_j y_3})^2 \geq 0$.

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Now we have $D_j = \mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_3 y_j})^2 = 0$, the well known Monge–Ampère equation

Solution to the Monge–Ampère equation

Theorem

(Pogorelov–1956) For $j = 1$ *or* 2, $\mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_3 y_j})^2 = 0$ *has the solution* $M(y) = y_j t_j + y_3 t_3 + t_0$ *on the characteristics* $y_j dt_j + y_3 dt_3 + dt_0 = 0$*, which are straight lines in the* $y_j \times y_3$ *plane. Furthermore,* t_0, t_j, t_3 *are constant on characteristics with the property* $M_{y_j} = t_j, M_{y_3} = t_3$.

Cases for characteristics

Because of the symmetry property of $\mathcal{M}$, we only need to consider the domain $\Xi_+ := \{y : -y_1 \leq y_2 \leq y_1, y_3 \geq 0, (y_1 - y_2)^p \leq y_3\}$ rather than Ξ.

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Strategy for finding explicit form of Bellman function

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4. Check to see if the solution satisfies weak concavity needed to be a Bellman function candidate.

Finding Bellman function candidate when $2 < p < \infty$

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For $2 < p < \infty$ and $|\tau| \leq \frac{1}{2}$ the solution to the Monge–Ampère equation is given by

$$M(y) = (1+\tau^2)^{\frac{p}{2}}[y_1^2 + 2\gamma y_1 y_2 + y_2^2]^{\frac{p}{2}} + ((p-1)^2+\tau^2)^{\frac{p}{2}}[y_3 - (y_1 - y_2)^p]$$

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This still needs to be finished here but we are confident that there will still be a counterexample in the general case.

Main Result

Goal: For $|\tau| \leq \frac{1}{2}$, prove that for all $n \in \mathbb{Z}_+$ and any $\{d_k\}_k$ martingale difference,

$$\left\| \sum_{k=1}^{n} \begin{pmatrix} \varepsilon_k \\ \tau \end{pmatrix} d_k \right\|_{L^p([0,1),\mathbb{R}^2)} \leq C_{p,\tau} \left\| \sum_{k=1}^{n} d_k \right\|_{L^p([0,1),\mathbb{R})},$$

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This estimate can be proven now that we have a Bellman function candidate.

To show that the sharp constant is $C_{p,\tau} = ((p^* - 1)^2 + \tau^2)^{\frac{1}{2}}$, we need to show that our Bellman candidate is actually the Bellman function by closing the door on Case (4).

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Again, once we close the door on Case (4), then for $|\tau| \leq \frac{1}{2}$, we will have $C_{p,\tau} = ((p^* - 1)^2 + \tau^2)^{\frac{1}{2}}$.

Thank you