## Directional Discrepancy

Dmitriy Bilyk<br>University of South Carolina, Columbia, SC

Fields Institute
Toronto, ON, Canada
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$\mathcal{P}_{N}$ - a set of $N$ points in $[0,1]^{d}$
$R \subset[0,1]^{d}$ - a measurable set.


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$$
D_{\Omega}(N)=\inf _{\mathcal{P}_{N}} \sup _{R \in \mathcal{A}_{\Omega}}\left|D\left(\mathcal{P}_{N}, R\right)\right|
$$

## No rotations: axis-parallel rectangles

- $\Omega=\{0\}$

Theorem (Lerch; Schmidt)

$$
D_{\Omega}(N) \approx \log N
$$

Theorem (L²: Roth; Davenport)

$$
D_{\Omega}^{2}(N) \approx \sqrt{\log N}
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- $\Omega$ finite $\longrightarrow$ same (Beck - Chen, Chen-Travaglini)


## All rotations

- $\Omega=S^{1}$


## Theorem (J. Beck)

$$
N^{1 / 4} \lesssim D_{\Omega}(N) \lesssim N^{1 / 4} \sqrt{\log N}
$$


discrepancy about $\log n$

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discrepancy about $n^{1 / 4}$

## All rotations: Beck's method

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Discrepancy measure: $D(A)=\sum_{p \in \mathcal{P}_{N}} \delta_{p}(A)-N \cdot \operatorname{vol}\left(A \cap[0,1]^{2}\right)$

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D(A+x)=\int_{\mathbb{R}^{2}} \mathbf{1}_{A+x}(y) d D(y)
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Discrepancy measure: $D(A)=\sum_{p \in \mathcal{P}_{N}} \delta_{p}(A)-N \cdot \operatorname{vol}\left(A \cap[0,1]^{2}\right)$

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\begin{aligned}
D(A+x) & =\int_{\mathbb{R}^{2}} \mathbf{1}_{A+x}(y) d D(y) \\
& =\left(\mathbf{1}_{A} * D\right)(x)
\end{aligned}
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## All rotations: Beck's method

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- $\widehat{\Delta}_{A}(\xi)=\widehat{\mathbf{1}}_{A}(\xi) \cdot \widehat{D}(\xi)$
- $\left\|\Delta_{A}\right\|_{2}^{2}=\left\|\widehat{\Delta}_{A}\right\|_{2}^{2}=\int_{\mathbb{R}^{2}}\left|\widehat{\mathbf{1}}_{A}(\xi)\right|^{2} \cdot|\widehat{D}(\xi)|^{2} d \xi$


## Rotated Squares

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- IF we had $\left|\widehat{\mathbf{1}}_{A_{r}}(\xi)\right|^{2} \gtrsim \frac{r}{r_{0}} \cdot\left|\widehat{\mathbf{1}}_{A_{r_{0}}}(\xi)\right|^{2}$ for $r>r_{0}$


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- THEN we would have

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\left\|\Delta_{A_{1}}\right\|_{2} \gtrsim \sqrt{\frac{1}{1 / 2 \sqrt{N}}}\left\|\Delta_{A_{1 / 2 \sqrt{N}}}\right\|_{2} \approx N^{1 / 4}
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- BUT

$$
\left|\widehat{\mathbf{1}}_{A_{r}}(\xi)\right|^{2}=\left(\frac{\sin \left(\pi \xi_{1} r\right)}{\pi \xi_{1}}\right)^{2} \cdot\left(\frac{\sin \left(\pi \xi_{2} r\right)}{\pi \xi_{2}}\right)^{2}
$$

## $\left|\widehat{\mathbf{1}}_{A_{r}}(\xi)\right|^{2}=\left(\sin \left(\pi \xi_{1} r\right) /\left(\pi \xi_{1}\right)\right)^{2} \cdot\left(\sin \left(\pi \xi_{2} r\right) /\left(\pi \xi_{2}\right)\right)^{2}$


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$\left|\widehat{\mathbf{1}}_{A_{r}}(\xi)\right|^{2}$ for $r=0.5$

## Rotated Squares

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\text { - }\left|\widehat{\mathbf{1}}_{A_{r}}(\xi)\right|^{2}=\left(\frac{\sin \left(\pi \xi_{1} r\right)}{\pi \xi_{1}}\right)^{2} \cdot\left(\frac{\sin \left(\pi \xi_{2} r\right)}{\pi \xi_{2}}\right)^{2}
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- Let $A_{r} \rho$ denote the rotation of $A_{r}$

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\frac{1}{2 \pi} \int^{2 \pi} \frac{1}{r} \int^{2 r}\left|\hat{\mathbf{1}}_{A_{\rho}}(\xi)\right|^{2}
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- Let $A_{r, \theta}$ denote the rotation of $A_{r}$ by $\theta$

$$
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- Let $A_{r, \theta}$ denote the rotation of $A_{r}$ by $\theta$

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\omega_{r}(\xi)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{1}{r} \int_{r}^{2 r}\left|\widehat{\mathbf{1}}_{A_{\rho, \theta}}(\xi)\right|^{2} d \rho d \theta \approx \min \left\{r^{4}, \frac{r}{|\xi|^{3}}\right\}
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- Thus $\frac{\omega_{r}(\xi)}{\omega_{r_{0}}(\xi)} \gtrsim \frac{r}{r_{0}}$


## Irrational Lattice

## Example

Let $\alpha$ be an irrational number. Define $\mathcal{P}_{N}=\left\{\left(\frac{i}{N},\{i \alpha\}\right)\right\}_{i=0}^{N-1}$

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## $\alpha$ badly approximable

For any $p, q \in \mathbb{Z}$

$$
\left|\alpha-\frac{p}{q}\right| \gtrsim \frac{1}{q^{2}}
$$



## Irrational Lattice

## Example

Let $\theta \in[0, \pi]$. Denote by $\mathcal{P}_{N}^{\theta}$ the lattice $\frac{1}{\sqrt{N}} \mathbb{Z}^{2}$ rotated by $\theta$ and intersected with $[0,1]^{2}$.

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## Theorem

If $\tan \theta$ is badly approximable, i.e.

$$
\left|\tan (\theta)-\frac{p}{q}\right| \gtrsim \frac{1}{q^{2}}
$$

for all integer $p, q$, then the discrepancy of $\mathcal{P}_{N}^{\theta}$ with respect to axis-parallel rectangles satisfies

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## Davenport's lemma

- Finitely many directions


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## Lemma (Davenport; Cassels)

For any $\phi_{1}, \phi_{2}, \ldots, \phi_{n}$, there exists $\theta$ such that for all $j=1, \ldots, n$

$$
\left|\tan \left(\theta-\phi_{j}\right)-\frac{p}{q}\right| \gtrsim \frac{1}{q^{2}}
$$

for all $p \in \mathbb{Z}, q \in \mathbb{N}$.

## Lacunary directions

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## Lemma (D.B., X. Ma, J. Pipher, C. Spencer)

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for all $p \in \mathbb{Z}, q \in \mathbb{N}$.

## Lacunary of finite order

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for all $p \in \mathbb{Z}, q \in \mathbb{N}$.

## Minkowski dimension

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\left|\tan (\theta-\phi)-\frac{p}{q}\right| \gtrsim \frac{1}{q^{\frac{2}{(1-d)^{2}}+\varepsilon}}
$$

for all $p \in \mathbb{Z}, q \in \mathbb{N}$ and any $\varepsilon>0$.

## Proof of the lemmas

- We construct a sequence $\ldots I_{n-1} \supset I_{n} \supset \ldots$ with $\left|I_{n}\right| \rightarrow 0$ so that

$$
\begin{equation*}
\left|\tan (\theta-\phi)-\frac{p}{q}\right| \geq \frac{c(n)}{q^{2}} \tag{*}
\end{equation*}
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for all $\theta \in I_{n}, \phi \in \Omega$, and $R(n) \leq q \leq R(n+1)$

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- Upper Minkowski dimension d: $N \leq C_{\varepsilon}\left(\frac{1}{\delta}\right)^{d+\varepsilon}$


## Proof of the lemmas

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- Cover $\Omega$ with $N(n)$ intervals $B_{k}$ of length $\delta(n)$
- Assume $\left(\theta_{1}, \phi_{1}, p_{1}, q_{1}\right)$ and $\left(\theta_{2}, \phi_{2}, p_{2}, q_{2}\right)$ with $\phi_{i} \in B_{k}, \theta_{i} \in I_{n-1}, R(n) \leq q \leq R(n+1)$ violate $(*)$
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$$
\left|\frac{p_{1}}{q_{1}}-\frac{p_{2}}{q_{2}}\right| \leq \frac{2 c(n)}{R(n)^{2}}+C\left|I_{n-1}\right|+C \delta(n)
$$

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for all $\theta \in I_{n}, \phi \in \Omega$, and $R(n) \leq q \leq R(n+1)$

- Cover $\Omega$ with $N(n)$ intervals $B_{k}$ of length $\delta(n)$
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$$

- Then there is at most one $p / q \in B_{k}$ with $R(n) \leq q_{i} \leq R(n+1)$, which violates $(*)$


## Proof of the lemmas

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- Then the remainder contains an interval of length $\left|I_{n}\right|$


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- Finite: $R(n)=a^{n}, c(n)=c, \delta(n)=0, N(n)=N$
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- Lacunary: $R(n)=n^{n / 2} \log ^{n / 2} n, c(n)=\frac{c}{n^{2} \log ^{2} n}$

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c(n) \approx \log ^{-2} R(n)
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- $\frac{2 c(n)}{R(n)^{2}}+C\left|I_{n-1}\right|+C \delta(n) \leq \frac{1}{R(n+1)^{2}}$
- Minkowski dimension $d: R(n)=2^{a^{n}}$,

$$
c(n)=c 2^{-2 a^{n}\left(a^{2}-1\right)} \approx R(n)^{-\left(2 a^{2}-1\right)}, \text { where } a=\frac{1}{1-d-\varepsilon}
$$

## Generalization

- Let $N(x)$ be an upper bound of the minimal number of intervals of length $x$ needed to cover $\Omega$.


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## Lemma (D.B., X. Ma, J. Pipher, C. Spencer)

There exists $\theta$ such that for all $\phi \in \Omega$

$$
\left|\tan (\theta-\phi)-\frac{p}{q}\right| \gtrsim F^{-1}\left(F^{-1}\left(\frac{1}{q^{2}}\right)\right)
$$

for all $p \in \mathbb{Z}, q \in \mathbb{N}$.

## Erdös-Turan inequality

- For a sequence $\omega=\left(\omega_{n}\right) \subset[0,1]$ :

$$
D_{N}(\omega)=\sup _{x \subset[0,1]}\left|\#\left\{n \leq N: \omega_{n} \leq x\right\}-N x\right|
$$

## Theorem (Erdös-Turan)

For any sequence $\omega \subset[0,1]$ we have

$$
D_{N}(\omega) \lesssim \frac{N}{m}+\sum_{h=1}^{m} \frac{1}{h}\left|\sum_{n=1}^{N} e^{2 \pi i h \omega_{n}}\right|
$$

for all natural numbers $m$.

- If $\omega_{n}=\{n \alpha\}$

$$
\left|\sum_{n=1}^{N} e^{2 \pi i h n \alpha}\right| \leq \frac{2}{\left|e^{2 \pi i h \alpha}-1\right|}=\frac{1}{|\sin (\pi h \alpha)|}=\frac{1}{\sin (\pi\|h \theta\|)} \lesssim \frac{1}{\|h \alpha\|}
$$

## Exponential sum estimates

Let $\psi$ be a non-decreasing function on $\mathbb{R}_{+}$.
$\alpha \in \mathbb{R}$ is of type $<\psi$ if for all $q \in \mathbb{N}$ we have $q\|q \alpha\|>1 / \psi(q)$, i.e.

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\left|\alpha-\frac{p}{q}\right|>\frac{1}{q^{2} \cdot \psi(q)}
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\begin{gathered}
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\sum_{h=1}^{m} \frac{1}{h\|h \theta\|} \lesssim \psi(2 m) \log m+\sum_{h=1}^{m} \frac{\psi(2 h) \log h}{h}
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Finitely many directions: $\psi(q)=$ const

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\end{gathered}
$$

Minkowski dimension $d: \psi(q) \approx q^{\frac{2}{(1-d)^{2}}-2+\varepsilon}$

## Directional Discrepancy

$$
D_{\Omega}\left(\mathcal{P}_{N}^{\theta}\right) \lesssim D_{c \sqrt{N}}(\{n \cdot \tan (\theta-\phi)\})
$$

## Directional Discrepancy

## Theorem (DB, Ma, Pipher, Spencer)

- Lacunary directions $\Omega=\left\{2^{-n}\right\}_{n=1}^{\infty}$

$$
D_{\Omega}(N) \lesssim \log ^{3} N
$$

- Lacunary of order M

$$
D_{\Omega}(N) \lesssim \log ^{2 M+1} N
$$

- $\Omega$ has upper Minkowski dimension $0 \leq d<1$

$$
D_{\Omega}(N) \lesssim N^{\frac{1}{2}-\frac{1}{2(\tau+1)}+\varepsilon},
$$

where $\tau=\frac{2}{(1-d)^{2}}-2$.

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- but for $\Omega=[0,2 \pi)$,

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D_{\Omega}(N) \lesssim N^{1 / 4} \log ^{1 / 2} N
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- but for $\Omega=[0,2 \pi)$,

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$$

- So $(*)$ is interesting only for $\frac{1}{2}-\frac{1}{2(\tau+1)}<\frac{1}{4}$, i.e. $\tau<1$,

$$
d<1-\left(\frac{2}{3}\right)^{\frac{1}{2}} \approx 0.1835 \ldots
$$

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$$

- If $\tan \theta$ is of type $<q^{\tau}$ for some $\tau \geq 0$,
i.e. for all $q \in \mathbb{N}$ we have $q \cdot\|q \cdot \tan \theta\| \gtrsim 1 / q^{\tau}$ or

$$
\left|\tan \theta-\frac{p}{q}\right| \gtrsim \frac{1}{q^{2+\tau}}
$$

then $\mathcal{P}_{N}^{\theta}$ satisfies

$$
D\left(\mathcal{P}_{N}^{\theta}\right) \gtrsim N^{\frac{1}{2}-\frac{1}{2(\tau+1)}-\varepsilon}
$$

for axis-parallel rectangles.

## Directional Discrepancy $L^{2}$

## Theorem (DB, Ma, Pipher, Spencer)

Let $\mu$ be any probability measure on $\mathcal{A}_{\Omega}$. Then

1) If $\Omega$ is a lacunary sequence, there exists $\mathcal{P} \subset[0,1]^{2}, \# \mathcal{P}=N$

$$
\left(\int_{\mathcal{A}_{\Omega}}\left|D_{\Omega}(\mathcal{P}, R)\right|^{2} d \mu(R)\right)^{\frac{1}{2}} \lesssim \log ^{\frac{5}{2}} N .
$$

2) If $\Omega$ is lacunary of order $M$, there exists $\mathcal{P} \subset[0,1]^{2}$, \#P $=N$

$$
\left(\int_{\mathcal{A}_{\Omega}}\left|D_{\Omega}(\mathcal{P}, R)\right|^{2} d \mu(R)\right)^{\frac{1}{2}} \lesssim \log ^{2 M+\frac{1}{2}} N .
$$

3) If $\Omega$ has upper Minkowski dimension $0 \leq d<1$, there exists $\mathcal{P}$

$$
\left(\int_{\mathcal{A}_{\Omega}}\left|D_{\Omega}(\mathcal{P}, R)\right|^{2} d \mu(R)\right)^{\frac{1}{2}} \lesssim N^{\frac{1}{2}-\frac{1}{2(\tau+1)}+\varepsilon}
$$

for any $\varepsilon>0$, if $\tau=\frac{2}{(1-d)^{2}}-2<1$

## Directional Discrepancy $L^{2}$

## Lemma

Let I be a finite interval of consecutive integers.

1) Assume $\tan \phi$ satisfies $\nu\|\nu \tan \phi\|>\frac{c}{\log ^{2 M} \nu}$, for all $\nu \in \mathbb{N}$. Then

$$
\sum_{\nu=1}^{\infty} \frac{1}{\nu^{2}}\left|\sum_{n \in I} e^{-2 \pi i \nu n \tan \phi}\right|^{2} \lesssim \log ^{4 M+1}|I|
$$

2) Assume $\tan \phi$ satisfies $\nu\|\nu \tan \phi\|>c \nu^{-\tau+\varepsilon}$, for all $\varepsilon>0$, where $0 \leq \tau<1$. Then

$$
\sum_{\nu=1}^{\infty} \frac{1}{\nu^{2}}\left|\sum_{n \in I} e^{-2 \pi i \nu n \tan \phi}\right|^{2} \lesssim|I|^{\frac{2 \tau}{\tau+1}+\varepsilon^{\prime}}, \text { where } \varepsilon^{\prime}=\mathcal{O}(\varepsilon)
$$

