Incomplete-market equilibria with exponential utilities

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GE - an exchange-economy model

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   1.1 Commodity space: $\mathbb{R}_+^C$, $C \in \mathbb{N}$,

   1.2 Agents (households, consumers): $i = 1, \ldots, I$, $I \in \mathbb{N}$,

      1.2.1 a preference relation $\preceq^i$,

      1.2.2 an initial endowment $e^i \in \mathbb{R}_{++}^C$,

2. Price systems: $p \in P^C$, where $P^C = \{p \in \mathbb{R}_+^C : |p|_{L^1} = 1\}$.

3. Budgets:

$$B^i(p) = \{x \in \mathbb{R}_+^C : p \cdot x \leq p \cdot e^i\},$$

4. Demand correspondences: (assumed to be functions)

$$\Delta^i(p) = \{x \in \mathbb{R}_+^C : y \preceq^i x, \\ \forall\, y \in B^i(p)\},$$

5. $p^* \in P^C$ is an equilibrium price if

$$\sum_{i=1}^{I} \Delta^i(p^*) = 0.$$

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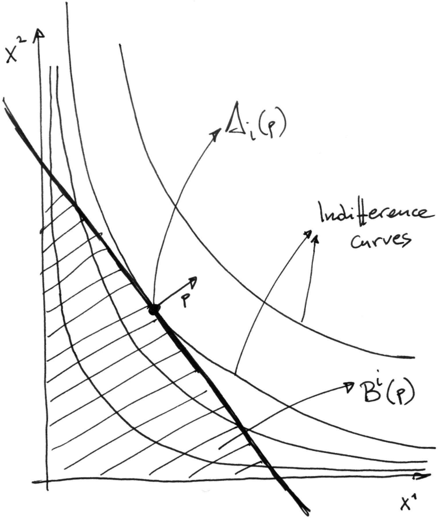

History + results

Some history

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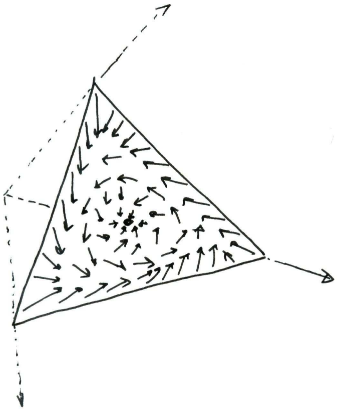
Classical results

1. Under minimal regularity assumptions, an equilibrium p^* always exists.
2. In general, the set of all equilibria is not even countable.
3. Generically (over all economies, appropriately parametrized by a subset of $\mathbb{R}^n$), the set of equilibria is finite.
4. Every equilibrium allocation is Pareto optimal.



The GEI model

Incompleteness

1. A budget set of the form $B^i(p) = \{x \in \mathbb{R}_+^C : p \cdot x \leq p \cdot e^i\}$ implies that

   "If it's affordable, it's available."

2. When transfer of the "consumption good" is possible only through a system of markets, not every transaction is implementable in general.
3. The generalization of the GE model - called the GEI model - deals with this case.

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Results

1. Equilibria exist only generically,
2. Generic local uniqueness may fail,
3. Equilibrium allocations are generically not Pareto optimal.

The stochastic model

Information

A filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ (where $\mathbb{P}$ is used only to fix the family of negligible sets).

Agents

$I \in \mathbb{N}$ agents, each of which is characterized by

1. a random endowment $\mathcal{E}^i \in \mathcal{F}_T$,
2. a utility function $U : \mathrm{Dom}(U) \to \mathbb{R}$, ($\mathrm{Dom}(U) = \mathbb{R}$ or $\mathrm{Dom}(U) = (0,\infty)$)
3. a subjective probability measure $\mathbb{P}^i \sim \mathbb{P}$.

(Note: 2. and 3. define the preference relation $\preceq^i$ by

$$X \preceq^i Y \;\Leftrightarrow\; \mathbb{E}^{\mathbb{P}^i}[U^i(X)] \le \mathbb{E}^{\mathbb{P}^i}[U^i(Y)],$$

in the manner of Alt, von Neumann and Morgenstern.)

Completeness constraints

A set $\mathcal{S}$ of $\{\mathcal{F}_t\}_{t\in[0,T]}$-semimartingales (possibly several-dimensional): the allowed asset-price dynamics.

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The equilibrium problem

(For simplicity, assume that a numéraire asset $\{B_t\}_{t\in[0,T]}$, with $B_t \equiv 1$, always exists.)

Problem

Does there exist $S \in \mathcal{S}$ such that

$$\sum_{i\in I} \hat{\pi}^i_t(S) = 0, \ \textit{for all } t \in [0,T], \textit{ a.s,}$$

where $\hat{\pi}^i(S) = \mathrm{argmax}_{\pi\in Adm}\, \mathbb{E}^{\mathbb{P}^i}[U^i(\mathcal{E}^i + \int_0^T \pi_u\, dS_u)]$ denotes the optimal trading strategy for the agent i when the market dynamics is given by S, and Adm is an appropriate admissibility set.

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Problem

It such an S exists and is unique, is it stable with respect to perturbations in the problem primitives?

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3. Information-constrained markets. Let $\{\mathcal{G}_t\}_{t\in[0,T]}$ be a sub-filtration of $\{\mathcal{F}_t\}_{t\in[0,T]}$, and let $\mathcal{S}$ be the class of all $\{\mathcal{G}_t\}_{t\in[0,T]}$-semimartingales.

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6. "Marketed-Set Constrained" markets Let V be a subspace of $\mathbb{L}^0(\mathcal{F}_T)$, and let $\mathcal{S}$ be the collection of all finite dimensional semimartingales $\{S_t\}_{t\in[0,T]}$ such that

$$\{x + \int_0^T \pi_t \, dS_t \; : \; x \in \mathbb{R}, \pi \in Adm\} = V.$$

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7. Markets with "fast-and-slow" information. Let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be generated by two orthogonal martingales M^1 and M^2, and let $\mathcal{S}$ be the collection of all processes of the form
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   Examples:
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   - $M^1 = B$ (Brownian motion), $M^2 = N_t - t$ (1-jump compensated Poisson process) so that a "typical" element of $\mathcal{S}$ is given by
   $$S_t = \int_0^t \lambda_u \, du + B_t.$$
   M^2 models an unpredictable catastrophic rare event (credit event, natural disaster, etc.)

Analysis

Representative-agent/Direct analysis

Uses the fact that equilibrium allocations are Pareto optimal; works (essentially) only for complete markets.

Literature in continuous time:

- **Complete markets:** Bank, Cvitanić, Dana, Duffie, Huang, Karatzas, Lakner, Lehoczky, Malamud, Riedel, Shreve, Ž., etc.
- **"Incomplete" markets:** Basak and Cuoco *'98* (incompleteness from restrictions in stock-market participation, logarithmic utility), Cheridito, Horst, Hugonnier, Mueller, Munk, Pirvu, etc.

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Excess-demand approach

Introduced in the early 20th century. Only recently applied in continuous time (Anthropelos and Ž (2008, 2009), Ž (2009), Zhao and Ž (2009))

1. Establish good topological properties of the excess demand function $S \mapsto \sum_i \hat{\pi}^i(S)$, and then
2. use a suitable fixed-point-type theorem to show existence (Brouwer, KKM, degree-based, etc.)

A partial-equilibrium pricing model (Anthropelos and Ž (2008))

Primitives

1. Information: $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, usual conditions.

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3. Completeness constraints: Let $\{S^L_t\}_{t\in[0,T]}$ be a locally bounded finite-dimensional semimartingale (NFLVR), and let $(B_1, \dots B_n) \in (\mathbb{L}^\infty)^n$ be a "bundle" of contingent claims. For $p \in \mathbb{R}$ and $k = 1, \dots, n$ define the processes $S^k_t(p) = \begin{cases} p, & t < T, \\ B_k, & t = T \end{cases}$, and set $\mathcal{S} = \{(S^L, S^1(p), \dots, S^n(p)) : p \in \mathbb{R}\}$.

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Existence and uniqueness results

Let $\Delta^i(p) \in \mathbb{R}^n$ denote the optimal quantity of $(B_1, \dots, B_n) \in \mathbb{R}^n$ for the agent i in the market $S(p) = (S^L, S^1(p), \dots, S^n(p))$. Then,

1. Δ^i is the Legendre transform of the (buyer's, conditional) indifference price of $(B_1, \dots, B_n)$.
2. There exists a unique equilibrium $S = S(p^*) \in \mathcal{S}$ and p^* can be characterized as a minimum of a functional involving conditional indifference prices.

A QUESTION OF STABILITY

1. Let's generalize the exponential-utility assumption: each agent uses a convex measure of risk ρ^i with the acceptance set $\mathcal{A}^i = \{X \in \mathbb{L}^\infty(\mathcal{F}_T) : \rho^i(X) \leq 0\}$.
2. The results about existence and uniqueness of the equilibrium still hold.
3. Question: is the equilibrium stable? Would it change dramatically if we replaced ρ^i by a "nearby" $\hat{\rho}^i$, $i = 1, \dots, I$?

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KURATOWSKI CONVERGENCE

A sequence $\{C_n\}_{n\in\mathbb{N}}$ of closed subsets of $\mathbb{R}^d$ converges to $C \subseteq \mathbb{R}^d$ in the Kuratowski sense - denoted by $C_n \xrightarrow{K} C$ - if

$$\text{Ls } C_n \subseteq C \subseteq \text{Li } C_n, \tag{1}$$

$$\begin{aligned} \text{Li } C_n &= \{c \in \mathbb{R}^d \,:\, c = \lim c_k, c_k \in C_k, \text{eventually}\} \\ \text{Ls } C_n &= \{c \in \mathbb{R}^d \,:\, c = \lim c_k, c_k \in C_k, \text{infinitely often}\}. \end{aligned} \tag{2}$$

A good first-order intuition for regular-enough $\{C_n\}_{n\in\mathbb{N}}$: sets converge in the Kuratowski sense if their "boundaries converge pointwise".

A partial-equilibrium pricing model (Anthropelos and Ž (2009)) II

A stability result

1. For an acceptance set $\mathcal{A}$, define its *B-projection* by

$$\mathcal{A}(B) = \Big\{ (m, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1} \ : \\ \exists\, \pi \in Adm,\ m + \alpha_1 B_1 + \dots + \alpha B_n + \int_0^T \pi_u \, dS^L_u \in \mathcal{A} \Big\}.$$

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2. Let $(\mathcal{A}_n^1, \dots, \mathcal{A}_n^I)$ be a sequence of I-tuples of acceptance sets such that for $i = 1, \dots, I$, we have
   - 2.1 $\mathcal{A}_n^i(B) \xrightarrow{K} \mathcal{A}^i(B)$, and
   - 2.2 the sets $\{\mathcal{A}_n^i(B)\}_{n \in \mathbb{N}}$ are uniformly strictly convex.
   - 2.3 a number of smaller, "non-triviality", assumptions,

   Then $p_n \to p$, where p_n is the unique equilibrium price for the setup with agent primitives $(\mathcal{A}_n^1, \dots, \mathcal{A}_n^I)$, and p is the unique equilibrium price for $(\mathcal{A}^1, \dots, \mathcal{A}^I)$

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   2.2 the sets $\{\mathcal{A}^i_n(B)\}_{n\in\mathbb{N}}$ are uniformly strictly convex.
   2.3 a number of smaller, "non-triviality", assumptions,

   Then $p_n \to p$, where p_n is the unique equilibrium price for the setup with agent primitives $(\mathcal{A}^1_n, \dots, \mathcal{A}^I_n)$, and p is the unique equilibrium price for $(\mathcal{A}^1, \dots, \mathcal{A}^I)$
3. Example: If all agents are exponential-utility maximizers, the condition $\gamma^i_n \to \gamma^i > 0$, $i = 1, \dots, I$, will do.

A partial-equilibrium pricing model (Anthropelos and Ž (2009)) II

A stability result

1. For an acceptance set $\mathcal{A}$, define its B-projection by

$$\mathcal{A}(B) = \Big\{ (m, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1} \ : \\ \exists\, \pi \in Adm,\ m + \alpha_1 B_1 + \dots + \alpha B_n + \int_0^T \pi_u \, dS_u^L \in \mathcal{A} \Big\}.$$

2. Let $(\mathcal{A}_n^1, \dots, \mathcal{A}_n^I)$ be a sequence of I-tuples of acceptance sets such that for $i = 1, \dots, I$, we have
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4. Example: if the acceptance sets/risk measures are utility-based, then it is (essentially) enough that $\mathbb{P}_n^i \to \mathbb{P}^i$ in total variation, $U_n^i \to U^i$ pointwise and $x_n^i \to x^i$ for the required Kuratowski convergence + uniform strict convexity.

A fast-and slow model (Ž (2009))

Primitives

1. Information: $(\Omega, \mathcal{F}, \{\mathcal{F}_t^{B,N}\}_{t\in[0,T]}, \mathbb{P})$, where $\{B_t\}_{t\in[0,T]}$ is a Brownian motion and $\{N_t\}_{t\in[0,T]}$ is a Poisson process with intensity μ stopped after the first jump.

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Restrict your search to only those D which admit the representation of the form

$$A_t = \int_0^t \lambda(u, B_u, N_{u-})\, du + B_t,$$

where $\lambda : [0,T] \times \mathbb{R} \times \{0,1\} \to \mathbb{R}$ belongs to the (anisotropic) Hölder space

$$C^\alpha\Big([0,T] \times \mathbb{R} \times \{0,1\}\Big).$$

(Note: C^α can be viewed as a parameter space for a convenient parametrization of the (subset of) the completeness constraint $\mathcal{S}$.)

A fast-and slow model (Ž (2009)) - the analysis

Step I

Express the agent i's optimal portfolio (we drop i from the notation) in the form

$$\hat{\pi}_t(\lambda) = \tfrac{1}{\gamma}\lambda(t, B_t, N_{t-}) - u_x(t, B_t, N_{t-}),$$

where u solves the semi-linear system of two interacting PDEs

$$\begin{cases} 0 = u_t + \tfrac{1}{2}u_{xx} - \lambda u_x + \tfrac{1}{2\gamma}\lambda^2 - \tfrac{\mu}{\gamma}\Big(\exp(-\gamma u_n) - 1\Big) \\ u(T, \cdot, \cdot) = g. \end{cases}$$

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Step II

Write the market-clearing condition $0 = \sum_{i=1}^{I} \hat{\pi}^i_t(\lambda)$ in the form $F(\lambda) = \lambda$, where

$$F(\lambda) = \bar{\gamma}\sum_{i=1}^{I} u^i_x(\lambda), \text{ and } \frac{1}{\bar{\gamma}} = \sum_{i=1}^{I}\frac{1}{\gamma^i}.$$

Step III

Show that the mapping

$$\lambda \mapsto u_x^i(\lambda)$$

is Lipschitz in C^α (use Schauder theory and Hölder interpolation).

(If you are curious, here is a (crude) estimate

$$L(R) = C\,T^{\frac{1+\alpha}{2+\alpha}} \exp\left(\exp\left(2 + 2\gamma^i ||g^i||_0 + TR^2 + 2\mu T\right)\right) \times \\ \times \left(||g^i||_{2+\alpha} + (1+T)(1+R^2)\right)^{6+4\alpha}.$$

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Pick T small enough and apply the Banach fixed-point theorem to the function F.

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Bonus

The fact that the Banach fixed-point theorem applies allows for efficient computation algorithms to be used.

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A parabolic system

It can be shown that $\lambda \in C^\alpha$ is an equilibrium market-price-of-risk if and only if $\lambda = \bar{\gamma} \sum_{i=1}^{I} u^i_x$, where $(u^1, \dots, u^I)$ solves the following quasilinear parabolic system of I equations:

$$\begin{cases} u^i_t + \frac{1}{2}\Delta u^i + \bar{\gamma} u^i_x \Big(\sum_{k=1}^{I} u^k_x \Big) + \frac{\bar{\gamma}^2}{2\gamma^i} \Big(\sum_{k=1}^{I} u^k_x \Big)^2 - \frac{\gamma^i}{2} \Big(u^i_y \Big)^2 = 0, \\ u^i(T,x,y) = g^i(x,y), \end{cases}$$

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A delicate analysis based on the use of Schauder's fixed-point theorem on an appropriately-tuned domain yields existence. Uniqueness follows from classical energy estimates.

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Backward SDEs

Once can rephrase the above in the language of BSDE: we obtain a coupled system of nonlinear BSDEs with quadratic growth. Existence can be obtained in fair generality. General uniqueness is still unavailable.