Finding Quantiles

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Based on Papers with Hanqing Jin, Zuoquan Xu (Oxford), and Xuedong He (Columbia)

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Portfolio Selection Models

Fundamental Assumptions

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- Economy: A linear pricing rule (existence of a pricing kernel)

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- Economy: A linear pricing rule (existence of a pricing kernel)
- Agent: "the more money (initial endowment) the better (w.r.t. her criterion)"

Expected Utility Maximisation

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\begin{array}{ll}
\underset{X}{\text{Max}} & Eu(X) \\
\text{Subject to} & E[\rho X] = x_0, \\
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Merton (1971); abundant research thereafter

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Markowitz (1952); abundant research thereafter

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Kulldorff (1993), Heath (1993), Browne (1999), Föllmer and Leukert (1999), Spivak and Cvitanić (1999), etc.

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Yaari (1987) - the criterion only

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- Risk averse iff T is convex (Yaari 1987)

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Lopes (1987) - criterion; Lopes and Oden (1999)- single period

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where $0 < w < 1, \ \ q_s, q_p > 0$

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- y^{q_s+1}: convex or risk averse (**security**)
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- An instantiation of behavioural economics/finance theory

Kahneman and Tversky's Prospect Theory

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Kahneman and Tversky (1979, 1991), Berkelaar, Kouwenberg and Post (2004), Jin and Zhou (2008)

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- Backbone of behavioral economics/finance theory

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  - Nonconcave in X: convex duality fails
  - Nonlinear expectation with Choquet integration: time-consistency or HJB fails

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YES WE CAN! All it takes: *A new perspective*

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- SP/A: $\int_0^\infty x d\left\{-T\left[1 - F_X(x)\right]\right\}$
- Prospect Theory: $\int_B^\infty u_+(x - B) d\left\{-T_+\left(1 - F_X(x)\right)\right\} - \int_{-\infty}^B u_-(B - x) d\left\{T_-\left(F_X(x)\right)\right\}$

Commonality in Six Models (Cont'd)

- **Commonality**: $\int_{-\infty}^{\infty} u(x) d\left[T(F_X(x))\right]$ where $u(\cdot)$ and $T : [0,1] \to [0,1]$ nonlinear (Quiggin 1982: rank dependent utility)

Commonality in Six Models (Cont'd)

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- ... by which we recover linear expectation and concavity (if $u(\cdot)$ is concave)!

Budget Constraint

- Express $E[\rho X] = x_0$ in terms of quantiles

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- Hence

$$E[\rho X] = x_0 \Leftrightarrow E\left[F_\rho^{-1}(1-Z)G(Z)\right] = x_0$$
$$\Leftrightarrow \int_0^1 F_\rho^{-1}(1-z)G(z)dz = x_0$$

A General Portfolio Selection Model

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- Also covers models involving VaR/CVaR objectives/constraints, mean–variance and many others
- Solvable by Lagrange!
- If $G^*(\cdot)$ is optimal then $X^* = G^*(1 - F_\rho(\rho))$: optimal terminal cash flow is anti-comonotonic w.r.t. pricing kernel ρ

Example 1. Goal Achieving

$$\begin{array}{ll} \text{Max} & P(X \geq b) \\ \text{s.t.} & E[\rho X] = x_0 \\ & X \geq 0 \end{array}$$

where $b > 0$, and $0 < x_0 < bE[\rho]$

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Theorem

(He and Zhou 2009) The unique optimal solution is $X^* = b\mathbf{1}_{(\rho\leq c^*)}$ *where* c^* *is such that* $E[\rho X^*] = x_0$. *The optimal value is* $F_\rho(c^*)$.

Example 2. Yaari's Model

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\begin{array}{ll}
\text{Max} & \int_0^\infty T\left(P(X > x)\right) dx \\
\text{s.t.} & E[\rho X] = x_0 \\
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where $T : [0,1] \longrightarrow [0,1]$, continuous, strictly increasing, C^1 on $(0,1)$, $T(0) = 0$, $T(1) = 1$

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Solution to Yaari's Model

Assumption

$M(z) := \frac{T'(1-z)}{F_\rho^{-1}(1-z)}$ *continuous on* $(0,1)$, *and* $\exists z_0 \in (0,1)$ *such that* $M(\cdot) \uparrow$ *on* $(0, z_0)$ *and* $\downarrow$ *on* $(z_0, 1)$.

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Remark

This assumption holds when ρ is lognormal and $T(z) = z^\gamma, \gamma > 1$.

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(He and Zhou 2009) The unique optimal solution is $X^* = b^* \mathbf{1}_{\rho \le c}$ where c is the unique root of

$$h(x) := xT(F_\rho(x)) - T'(F_\rho(x)) \int_0^x y dF_\rho(y)$$

on $(F_\rho^{-1}(1-z_0), \bar{\rho})$, and $b^* > 0$ is such that $E[\rho X^*] = x_0$.

Solution to SP/A Model

Assumption

(i) $u(\cdot)$ *strictly increasing, strictly concave, differentiable, and satisfies Inada condition.*

(ii) $\frac{F_\rho^{-1}(z)}{T'(z)}$ *decreasing on* $z \in (0,1)$.

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This assumption holds when ρ *is lognormal and* T *proposed by Lopes (1987).*

Solution to SP/A Model (Cont'd)

Theorem

(He and Zhou 2009) If $x_0 > AE[\rho \mathbf{1}_{\{\rho \le F^{-1}(\alpha)\}}]$, *then the unique optimal solution is*

$$X^* = (u')^{-1}\left(\frac{\lambda^* \rho}{T'(F_\rho(\rho))}\right)\mathbf{1}_{\{\rho \ge F^{-1}(\alpha)\}} + \left[(u')^{-1}\left(\frac{\lambda^* \rho}{T'(F_\rho(\rho))}\right) \vee A\right]\mathbf{1}_{\{\rho < F^{-1}(\alpha)\}}.$$

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Example 4. Prospect Model

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\text{Max} & \int_0^\infty T_+\left(P\left(u_+\left((X-B)_+\right)>x\right)\right)dx \\
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\text{s.t.} & E[\rho X]=x_0
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Optimal solution (Jin and Zhou 2008)

$$X^* = (u'_+)^{-1}\left(\frac{\lambda\rho}{T'_+(F_\rho(\rho))}\right)\mathbf{1}_{\rho\le c^*} - \frac{x_+^* - x_0}{E[\rho\mathbf{1}_{\rho>c^*}]}\mathbf{1}_{\rho>c^*}$$

Additional Model 1: Coherent Risk Measure

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\begin{array}{ll}
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Quantile Formulation

- Representation

$$C(X) = \sup_{\mu \in \mathcal{M}} \int_{[0,1]} AV@R_z(X)\mu(dz)$$

where $\mathcal{M}$ is a subset of probability measures on [0,1],
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- Explicit solution when C is comonotonic ($\mathcal{M}$ is a singleton)

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Quantile Formulation

- Representation

$$C(X) = \sup_{\mu \in \mathcal{M}} \int_{[0,1]} AV@R_z(X)\mu(dz)$$

where $\mathcal{M}$ is a subset of probability measures on [0,1], $AV@R_z(X) := \frac{1}{z}\int_0^z V@R_s(X)ds = -\frac{1}{z}\int_0^z G(s)ds$

- Minimax

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- He, Jin, and Zhou (2009)

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- A new paradigm for portfolio selection, and – hopefully – beyond

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