# WELLPOSEDNESS OF SECOND ORDER BSDEs

Nizar TOUZI

Ecole Polytechnique Paris

Joint with Mete SONER and Jianfeng ZHANG

Foundations of Financial Mathematics
Fields Institute, Toronto
January 11-15, 2010

# Outline

# Outline

# Hedging under Gamma constraints

- nonrisky asset $S^0 \equiv 1$ normalized to unity
- Risky asset defined by BS model $dS_t = S_t \sigma dW_t$
- Wealth process : $Y_t^Z = Y_0 + \int_0^t Z_u dS_u$, where $Z$ is the portfolio strategy assumed to be a semimartingale with

$$d\langle Z, S\rangle_t \;=\; \Gamma_t d\langle S\rangle_t \;\; \text{(the so-called Gamma)}$$

- Given a payoff $\xi \in \mathbb{L}^0(\mathcal{F}_T)$, superhedging problem :

$$V_0 \;:=\; \inf\left\{Y_0 : \;\exists\, Z \in \mathcal{Z},\; \underline{\Gamma} \le \Gamma \le \overline{\Gamma} \text{ and } Y_T^Z \ge \xi \;\mathbb{P}-\text{a.s.}\right\}$$

- Or the corresponding hedging problem : find $Z \in \mathcal{Z}$ and some "minimal" nondecreasing process $K$, $K_0 = 0$, such that

$$Y_T^Z - K_T = \xi,\; \mathbb{P}-\text{a.s.}$$

# Outline

# Market illiquidity Çetin, Jarrow and Protter 2004

$S^0 \equiv 1$ and Risky asset price is defined by

- the marginal price $S_t$, $t \geq 0$
- the supply curve $\nu \longmapsto \mathcal{S}(.,\nu)$with $\mathcal{S}(s,0) = s$ :

$$\mathcal{S}(S_t,\nu) \;\; \text{price per share of } \nu \text{ risky assets}$$

The self-financing condition leads to :

$$Y_T \quad = \quad Y_0 + \sum Z_{t_{i-1}} \Delta S_{t_i} - \sum \Delta Z_{t_i} \left[ \mathcal{S}\left(S_{t_i}, \Delta Z_{t_i}\right) - \mathcal{S}\left(S_{t_i}, 0\right) \right]$$

Assume $\nu \longmapsto \mathcal{S}(.,\nu)$ is smooth (unlike proportional transaction costs models), then :

$$Y_T = Y_0 + \int_0^T Z_t dS_t - \int_0^T \frac{\partial \mathcal{S}}{\partial \nu}\left(S_t, 0\right) d\langle Z^c \rangle_t - \sum_{t \leq T} \Delta Z_t \left[ \mathcal{S}\left(S_t, \Delta Z_t\right) - S_t \right]$$

# The Hedging Problem

- Let $dS_t = S_t \sigma dW_t$, then

$$d\langle Z^c\rangle_t = \Gamma_t^2 \, d\langle S\rangle_t : \qquad \text{the so-called Gamma...}$$

- Wealth process (jumps in portfolio are sub-optimal) :

$$dY_t = Z_t dS_t - \frac{\partial \mathcal{S}}{\partial \nu}(S_t, 0)\Gamma_t^2 \, d\langle S\rangle_t \quad \text{where} \quad d\langle Z, S\rangle_t =: \Gamma_t d\langle S\rangle_t$$

- For a contingent claim $\xi$, **Super-hedging problem**

$$V := \inf\left\{y \ : \ Y_T^{y,Z} \geq \xi \ \mathbb{P} - \text{a.s. for some "admissible" } Z\right\}$$

or, the corresponding hedging problem

$$Y_T^{y,Z} - K_T \quad = \quad \xi, \ \mathbb{P} - \text{a.s. for some "admissible" } Z$$

# Main difficulty

Without further restrictions on trading strategies, **the continuous-time problem reduces to Black-Scholes** !

**Lemma** (Bank-Baum 04) *For predictable $W-$integ. càdlàg process $\phi$, and $\varepsilon > 0$, there exists an absolutely continuous predictable process $\phi^\varepsilon_t = \phi^\varepsilon_0 + \int_0^t \alpha_r dr$* such that

$$\sup_{0\le t\le 1} \left| \int_0^t \phi_r dW_r - \int_0^t \phi^\varepsilon_r dW_r \right| \le \varepsilon$$

$\Longrightarrow$ If the "admissibility" set allows for arbitrary a.c. portfolio $Z_t = Z_0 + \int_0^t \alpha_u du$, then $V = V^{\mathrm{BS}}$ (with $\Gamma = 0$ !)

# Asymptotics of the discrete-time solution

**BUT** Gokay and Soner 09 showed that the discrete-time super-hedging cost (with time step $\frac{1}{n}$)

$$V^n \longrightarrow V^\infty \neq V^{\mathrm{BS}} \ !$$

$V^\infty$ is characterized as the unique viscosity solution of

$$-V_t^\infty(t,s) + \frac{1}{4}s^2\sigma(t,s)^2\ell(s)\left[1-\left(\left(\frac{V_{ss}^\infty(t,s)}{\ell(s)}+1\right)^+\right)^2\right] \quad = \quad 0$$

with $V^\infty(T,.) = g$ and $-C \le V^\infty \le C(1+s)$. Here $\ell := \left(4\frac{\partial \mathcal{S}}{\partial \nu}\right)^{-1}$

**In the continuous-time problem**, Cetin, Soner and T. derive directly this nonlinear PDE under appropriate restrictions on the trading strategies... similar to previous work on Gamma constraints Cheridito, Soner, T.

# Outline

1. Motivations
   - Hedging under Gamma constraints
   - Hedging under market illiquidity
   - Probabilistic numerics for nonlinear PDEs
2. Backward SDEs : a quick review
3. 2nd order backward SDEs
   - The CSTV framework
   - An alternative formulation of 2BSDEs
4. Main results
   - Representation, uniqueness
   - Existence
   - Connection with PDEs and stochastic control

# Intuitive introduction of the scheme

- Isolate a diffusion part in the equation :

$$0 \;=\; -v_t(t,x) - \frac{1}{2}1\Delta v(t,x) - f\left(x, Dv(t,x), D^2 v(t,x)\right)$$

- The Monte Carlo component  Let $X_s = x + 1W_{s-t+h}$, $s \geq t-h$, evaluate at $(s, X_s)$, and take expectations :

$$\begin{aligned} 0 \;&=\; \mathbb{E}\left[\int_{t-h}^{t} -(v_t + \frac{1}{2}\Delta v)(s,X_s)ds - \int_{t-h}^{t} f\left(., Dv, D^2\right)(s,X_s)ds\right] \\ &=\; v(t-h,x) - \mathbb{E}\left[v(t,X_t) + \int_{t-h}^{t} f\left(., Dv, D^2\right)(s,X_s)ds\right] \end{aligned}$$

- The Finite Differences component  Natural approximation

$$\hat{v}(t-h,x) = \mathbb{E}\left[\hat{v}(t,X_t)\right] + h\, f\left(x, \mathbb{E}[D\hat{v}(t,X_t)], \mathbb{E}[D^2\hat{v}(t,X_t)]\right)$$

To get rid of differentiation, apply an ibp argument

# Intuitive introduction of the scheme

- Isolate a diffusion part in the equation :

$$0 = -v_t(t,x) - \frac{1}{2}1\Delta v(t,x) - f\left(x, Dv(t,x), D^2v(t,x)\right)$$

- The Monte Carlo component Let $X_s = x + 1W_{s-t+h}$, $s \geq t-h$, evaluate at $(s, X_s)$, and take expectations :

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_{t-h}^{t} -(v_t + \frac{1}{2}\Delta v)(s, X_s)ds - \int_{t-h}^{t} f\left(., Dv, D^2\right)(s, X_s)ds\right] \\ &= v(t-h, x) - \mathbb{E}\left[v(t, X_t) + \int_{t-h}^{t} f\left(., Dv, D^2\right)(s, X_s)ds\right] \end{aligned}$$

- The Finite Differences component Natural approximation

$$\hat{v}(t-h,x) = \mathbb{E}\left[\hat{v}(t,X_t)\right] + h\,f\left(x, \mathbb{E}[D\hat{v}(t,X_t)], \mathbb{E}[D^2\hat{v}(t,X_t)]\right)$$

To get rid of differentiation, apply an ibp argument

# Intuitive introduction of the scheme

- Isolate a diffusion part in the equation :

$$0 = -v_t(t,x) - \frac{1}{2}1\Delta v(t,x) - f\left(x, Dv(t,x), D^2v(t,x)\right)$$

- The Monte Carlo component   Let $X_s = x + 1W_{s-t+h}$, $s \geq t-h$, evaluate at $(s, X_s)$, and take expectations :

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_{t-h}^{t} -(v_t + \frac{1}{2}\Delta v)(s, X_s)ds - \int_{t-h}^{t} f\left(., Dv, D^2\right)(s, X_s)ds\right] \\ &= v(t-h, x) - \mathbb{E}\left[v(t, X_t) + \int_{t-h}^{t} f\left(., Dv, D^2\right)(s, X_s)ds\right] \end{aligned}$$

- The Finite Differences component   Natural approximation

$$\hat{v}(t-h, x) = \mathbb{E}\left[\hat{v}(t, X_t)\right] + h\, f\left(x, \mathbb{E}[D\hat{v}(t, X_t)], \mathbb{E}[D^2\hat{v}(t, X_t)]\right)$$

To get rid of differentiation, apply an ibp argument

# Intuitive introduction of the scheme

- Isolate a diffusion part in the equation :

$$0 = -v_t(t,x) - \frac{1}{2} 1 \Delta v(t,x) - f\left(x, Dv(t,x), D^2 v(t,x)\right)$$

- The Monte Carlo component Let $X_s = x + 1 W_{s-t+h}$, $s \geq t-h$, evaluate at $(s, X_s)$, and take expectations :

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_{t-h}^{t} -(v_t + \frac{1}{2}\Delta v)(s,X_s)ds - \int_{t-h}^{t} f\left(., Dv, D^2\right)(s,X_s)ds\right] \\ &= v(t-h,x) - \mathbb{E}\left[v(t,X_t) + \int_{t-h}^{t} f\left(., Dv, D^2\right)(s,X_s)ds\right] \end{aligned}$$

- The Finite Differences component Natural approximation

$$\hat{v}(t-h,x) = \mathbb{E}\left[\hat{v}(t,X_t)\right] + h\, f\left(x, \mathbb{E}[D\hat{v}(t,X_t)], \mathbb{E}[D^2\hat{v}(t,X_t)]\right)$$

To get rid of differentiation, apply an ibp argument

# Intuition From Greeks Calculation

- Using the approximation $f'(x) \sim_{h=0} \mathbb{E}[f'(x+W_h)]$ :

$$\begin{aligned} f'(x) &\sim \int f'(x+y)\frac{e^{-y^2/(2h)}}{\sqrt{2\pi}}dy \\ &= \int f(x+y)\frac{y}{h}\frac{e^{-y^2/(2h)}}{\sqrt{2\pi}}dy \\ &= \mathbb{E}\left[f(x+W_h)\frac{W_h}{h}\right] \end{aligned}$$

- Similarly, by an additional integration by parts :

$$\begin{aligned} f''(x) &= \int f(x+y)\frac{y^2-h}{h^2}\frac{e^{-y^2/(2h)}}{\sqrt{2\pi}}dy \\ &= \mathbb{E}\left[f(x+W_h)\left(\frac{W_h^2-h}{h^2}\right)\right] \end{aligned}$$

# A probabilistic numerical scheme for fully nonlinear PDEs

This suggests the following discretization :

$$
\begin{aligned}
Y^n_{t_n} &= g\left(X^n_{t_n}\right) \ , \\
Y^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\right] + f\left(X^n_{t_{i-1}}, Y^n_{t_{i-1}}, Z^n_{t_{i-1}}, \Gamma^n_{t_{i-1}}\right)\Delta t_i \ , \quad 1 \le i \le n \ , \\
Z^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i} \frac{\Delta W_{t_i}}{\Delta t_i}\right] \\
\Gamma^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i} \frac{|\Delta W_{t_i}|^2 - \Delta t_i}{|\Delta t_i|^2}\right]
\end{aligned}
$$

Convergence results :

- $f_\gamma = f_z = 0$ : Bally, Pagès 2003
- $f_\gamma = 0$ : Bouchard, T. (2004) and Zhang (2004)
- General : Fahim,T.,Warin (2009)

# A probabilistic numerical scheme for fully nonlinear PDEs

This suggests the following discretization :

$$
\begin{aligned}
Y^n_{t_n} &= g\left(X^n_{t_n}\right) ,\\
Y^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\right] + f\left(X^n_{t_{i-1}}, Y^n_{t_{i-1}}, Z^n_{t_{i-1}}, \Gamma^n_{t_{i-1}}\right)\Delta t_i\ , \quad 1 \le i \le n ,\\
Z^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\frac{\Delta W_{t_i}}{\Delta t_i}\right]\\
\Gamma^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\frac{|\Delta W_{t_i}|^2 - \Delta t_i}{|\Delta t_i|^2}\right]
\end{aligned}
$$

Convergence results :

- $f_\gamma = f_z = 0$ : Bally, Pagès 2003
- $f_\gamma = 0$ : Bouchard, T. (2004) and Zhang (2004)
- General : Fahim,T.,Warin (2009)

# Connection with BSDEs

Convergence of the process $(Y_{t_i}, Z_{t_i}, \Gamma_{t_i})$

- $f_\gamma = 0$ : convergence towards the solution $(Y, Z)$ of the BSDE :

$$dY_t = -f(t, X_t, Y_t, Z_t)dW_t + Z_t dW_t, \qquad Y_T = g(X_T), \; \mathbb{P} - \text{a.s.}$$

- fully nonlinear ? ? ? we expect

$$dY_t = -f(t, X_t, Y_t, Z_t, \Gamma_t)dW_t + Z_t dW_t, \qquad Y_T = g(X_T), \; \mathbb{P} - \text{a.s.}$$

where $\Gamma_t dt := d\langle Z, W\rangle_t$

# Outline

# Backward SDEs

Pardoux and Peng (1990, 1992) : $W$ BM on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \mathbb{F}^W$

• For $\xi \in \mathbb{L}^2$, $H_t(y,z)$ Lipschitz in $(y,z)$, $H_\cdot(0,0) \in \mathbb{H}^2$ the BSDE

$$dY_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \qquad Y_1 = \xi$$

has a unique solution $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2$

• For $H \equiv 0$, this is just the martingale representation theorem

• Easy proof by means of a fixed point argument

• Other (important) extensions : obstacle, quadratic in $z$, multidimensional $Y$, ...

## Backward SDEs

Pardoux and Peng (1990, 1992) : $W$ BM on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \mathbb{F}^W$

• For $\xi \in \mathbb{L}^2$, $H_t(y,z)$ Lipschitz in $(y,z)$, $H_.(0,0) \in \mathbb{H}^2$ the BSDE

$$dY_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \qquad Y_1 = \xi$$

has a unique solution $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2$

• For $H \equiv 0$, this is just the martingale representation theorem

• Easy proof by means of a fixed point argument

• Other (important) extensions : obstacle, quadratic in $z$, multidimensional $Y$, ...

# Backward SDEs

Pardoux and Peng (1990, 1992) : $W$ BM on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \mathbb{F}^W$

- For $\xi \in \mathbb{L}^2$, $H_t(y, z)$ Lipschitz in $(y, z)$, $H_.(0, 0) \in \mathbb{H}^2$ the BSDE

$$dY_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \qquad Y_1 = \xi$$

has a unique solution $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2$

- For $H \equiv 0$, this is just the martingale representation theorem

- Easy proof by means of a fixed point argument

- Other (important) extensions : obstacle, quadratic in $z$, multidimensional $Y$, ...

# Backward SDEs

Pardoux and Peng (1990, 1992) : $W$ BM on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \mathbb{F}^W$

• For $\xi \in \mathbb{L}^2$, $H_t(y,z)$ Lipschitz in $(y,z)$, $H_.(0,0) \in \mathbb{H}^2$ the BSDE

$$dY_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \qquad Y_1 = \xi$$

has a unique solution $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2$

• For $H \equiv 0$, this is just the martingale representation theorem

• Easy proof by means of a fixed point argument

• Other (important) extensions : obstacle, quadratic in $z$, multidimensional $Y$, ...

# Connection with PDEs

In the previous context of the BSDE :

$$dY_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \qquad Y_1 = \xi$$

• Assume further that $H_t(y,z) = h(t, X_t, y, z)$, $\xi = g(X_1)$, and

$$dX_t \quad = \quad b(t, X_t)dt + \sigma(t, X_t)dW_t$$

Then $Y_t = V(t, X_t)$ for some deterministic measurable function $V$

• $V$ is a viscosity solution of the semilinear PDE

$$\partial_t V + \frac{1}{2}\sigma^2 D^2 V + bDV + h(t, x, V, \sigma DV) = 0, \quad V(1, x) = g(x)$$

# Connection with PDEs

In the previous context of the BSDE :

$$dY_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \qquad Y_1 = \xi$$

• Assume further that $H_t(y,z) = h(t, X_t, y, z)$, $\xi = g(X_1)$, and

$$dX_t \;=\; b(t, X_t)dt + \sigma(t, X_t)dW_t$$

Then $Y_t = V(t, X_t)$ for some deterministic measurable function $V$
• $V$ is a viscosity solution of the semilinear PDE

$$\partial_t V + \frac{1}{2}\sigma^2 D^2 V + bDV + h(t, x, V, \sigma DV) = 0, \quad V(1,x) = g(x)$$

# Motivation from finance

The BSDE

$$dY_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \qquad Y_1 = \xi$$

appeared in many contexts :

- Classical hedging problem in finance ($H \equiv 0$)
- Hedging under different lending and borrowing rates, hedging under portfolio constraints (+ nondecreasing process),
- Recursice utility, Risk measures/monetary utility functions
- Portfolio optimization (only in exponential or power expected utility framework)

# Outline

# Outline

# Second Order Backward SDEs

Cheridito, Soner, T. and Victoir 07 (CSTV) :

• 2BSDE :

$$dY_t \;=\; -H_t(Y_t, Z_t, \Gamma_t)dt + Z_t \circ dW_t, \quad \Gamma_t dt := d\langle Z, W\rangle_t, \quad Y_1 = \xi$$

where

$$Z_t \circ dW_t = Z_t dW_t + \frac{1}{2} d\langle Z, W\rangle_t = Z_t dW_t + \frac{1}{2}\Gamma_t dt$$

is the Fisk-Stratonovich stochastic integration.

• If $H_t = h(t, W_t, Y_t, Z_t, \Gamma_t)$ and $\xi = g(W_1)$, then $Y_t = V(t, W_t)$, where $V$ is associated with the fully nonlinear PDE :

$$\partial_t V + h(t, x, V, DV, D^2 V) = 0 \quad \text{and} \quad V(1, x) = g(x).$$

# Second Order Backward SDEs

Cheridito, Soner, T. and Victoir 07 (CSTV) :

• 2BSDE :

$$dY_t \;=\; -H_t(Y_t, Z_t, \Gamma_t)dt + Z_t \circ dW_t, \quad \Gamma_t dt := d\langle Z, W\rangle_t, \quad Y_1 = \xi$$

where

$$Z_t \circ dW_t = Z_t dW_t + \frac{1}{2} d\langle Z, W\rangle_t = Z_t dW_t + \frac{1}{2}\Gamma_t dt$$

is the Fisk-Stratonovich stochastic integration.

• If $H_t = h(t, W_t, Y_t, Z_t, \Gamma_t)$ and $\xi = g(W_1)$, then $Y_t = V(t, W_t)$, where $V$ is associated with the fully nonlinear PDE :

$$\partial_t V + h(t, x, V, DV, D^2 V) = 0 \quad \text{and} \quad V(1, x) = g(x).$$

# The uniqueness result of CSTV (only Markov case)

- Existence : if corresponding PDE has a smooth solution, then

$$Y_t = V(t, W_t), \quad Z_t = DV(t, W_t), \quad \Gamma_t = D^2 V(t, W_t).$$

- Uniqueness : Second Order Stochastic Target Problem

$$\begin{aligned} V(t,x) &:= \inf\left\{y : \; Y_1^{y,Z} \ge g(W_1) \text{ for some } Z \in \mathcal{Z}\right\} \\ U(t,x) &:= \sup\left\{y : \; Y_1^{y,Z} \le g(W_1) \text{ for some } Z \in \mathcal{Z}\right\} \end{aligned}$$

$V$ and $U$ are resp. viscosity super and subsolution of the PDE

If the comparison principle for viscosity solutions of PDE holds, then 2BSDE has a unique solution in class $\mathcal{Z}$

# The uniqueness result of CSTV (only Markov case)

- Existence : if corresponding PDE has a smooth solution, then

$$Y_t = V(t, W_t), \quad Z_t = DV(t, W_t), \quad \Gamma_t = D^2 V(t, W_t).$$

- Uniqueness : Second Order Stochastic Target Problem

$$\begin{aligned} V(t,x) &:= \inf\left\{ y : \ Y_1^{y,Z} \geq g(W_1) \text{ for some } Z \in \mathcal{Z} \right\} \\ U(t,x) &:= \sup\left\{ y : \ Y_1^{y,Z} \leq g(W_1) \text{ for some } Z \in \mathcal{Z} \right\} \end{aligned}$$

$V$ and $U$ are resp. viscosity super and subsolution of the PDE

If the comparison principle for viscosity solutions of PDE holds, then 2BSDE has a unique solution in class $\mathcal{Z}$

# The admissibility set $\mathcal{Z}$ in CSTV

**Definition** $Z \in \mathcal{Z}$ if it is of the form

$$Z_t = \sum_{n=0}^{N-1} z_n \mathbf{1}_{\{t<\tau_{n+1}\}} + \int_0^t \alpha_s ds + \int_0^t \Gamma_s dW_s$$

• $(\tau_n)$ is an $\nearrow$ seq. of stop. times, $z_n$ are $\mathcal{F}_{\tau_n}-$measurable, $\|N\|_\infty < \infty$
• $Z_t$ and $\Gamma_t$ are $\mathbb{L}^\infty-$bounded up to some polynomial of $X_t$
• $\Gamma_t = \Gamma_0 + \int_0^t a_s ds + \int_0^t \xi_s dW_s$, $0 \le t \le T$, and

$$\|\alpha\|_{B,b} + \|a\|_{B,b} + \|\xi\|_{B,2} < \infty, \qquad \|\phi\|_{B,b} := \left\| \sup_{0\le t\le T} \frac{|\phi_r|}{1+X_t^B} \right\|_{\mathbb{L}^b}$$

# Uniqueness in larger class

**Counter-example** The following linear 2BSDE with constant coefficients has a nonzero solution in $\mathbb{L}^2$ :

$$dY_t = -\frac{1}{2}c\Gamma_t dt + Z_t \circ dW_t, \qquad Y_1 = 0,$$

whenever $c \neq 1$

# Outline

1. Motivations
   - Hedging under Gamma constraints
   - Hedging under market illiquidity
   - Probabilistic numerics for nonlinear PDEs
2. Backward SDEs : a quick review
3. 2nd order backward SDEs
   - The CSTV framework
   - An alternative formulation of 2BSDEs
4. Main results
   - Representation, uniqueness
   - Existence
   - Connection with PDEs and stochastic control

## General framework

$\Omega := C([0,1],\mathbb{R}^d)$, $B$ : coordinate process, $\mathbb{P}_0$ : Wiener measure
$\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le 1}$ : filtration generated by $B$, $\mathbb{F}^+$

$\mathbb{P}$ is a local martingale measure if $B$ local martingale under $\mathbb{P}$

Karandikar 95 : $\int_0^t B_s dB_s$, defined $\omega-$wise, and coincides with the Itô integral, $\mathbb{P}-$a.s. for all local martingale measure $\mathbb{P}$. Then

$$\langle B\rangle_t := B_t B_t^{\mathrm{T}} - 2\int_0^t B_s dB_s^{\mathrm{T}} \quad \text{and} \quad \hat{a}_t := \overline{\lim_{\varepsilon\downarrow 0}} \frac{1}{\varepsilon}\Big(\langle B\rangle_t - \langle B\rangle_{t-\varepsilon}\Big),$$

defined $\omega-$wise

$\overline{\mathcal{P}}_W$ :set of all local martingale measures $\mathbb{P}$ such that

$$\langle B\rangle_t \quad \text{is a. c. in } t \text{ and} \quad \hat{a} \text{ takes values in } \mathbb{S}_d^{>0}(\mathbb{R}),\ \mathbb{P}-\text{a.s.}$$

# General framework, continued

For every $\mathbb{F}-$prog. meas. $\alpha$ valued in $\mathbb{S}_d^{>0}(\mathbb{R})$ with $\int_0^1 |\alpha_t|dt < \infty$, $\mathbb{P}_0-$a.s. Define

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \quad \text{where} \quad X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, t \in [0,1], \mathbb{P}_0 - \text{a.s.}$$

$\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$ : collection of all such $\mathbb{P}^\alpha$

Then every $\mathbb{P} \in \overline{\mathcal{P}}_S$

- satisfies the Blumenthal zero-one law
- and the martingale representation property

# Nonlinear generators

$H_t(\omega, y, z, \gamma) : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \longrightarrow \mathbb{R}$

$D_H \subset \mathbb{R}^{d\times d}$ given, containing 0

- For fixed $(y, z, \gamma)$, $H$ is $\mathbb{F}-$progressively measurable
- $H$ is uniformly Lipschitz in $(y, z)$, lsc in $\gamma$
- $H$ is uniformly continuous in $\omega$ under the $\mathbb{L}^\infty-$norm

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} a : \gamma - H_t(\omega, y, z, \gamma) \right\}, \quad a \in \mathbb{S}_d^{>0}(\mathbb{R});$$

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \quad \text{and} \quad \hat{F}_t^0 := \hat{F}_t(0, 0)$$

$\mathcal{P}_H$ : set of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that

$$\underline{a}_{\mathbb{P}} \leq \hat{a} \leq \overline{a}_{\mathbb{P}}, \text{ for some } \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \text{ and } \mathbb{E}^{\mathbb{P}}\left[\int_0^1 |\hat{F}_t^0|^2 dt\right] < \infty.$$

**Def** $\mathcal{P}_H-$q.s. means $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_H$ (Denis-Martini 04)

# Nonlinear generators

$H_t(\omega, y, z, \gamma) : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$

$D_H \subset \mathbb{R}^{d\times d}$ given, containing 0

- For fixed $(y, z, \gamma)$, $H$ is $\mathbb{F}-$progressively measurable
- $H$ is uniformly Lipschitz in $(y, z)$, lsc in $\gamma$
- $H$ is uniformly continuous in $\omega$ under the $\mathbb{L}^\infty-$norm

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} a : \gamma - H_t(\omega, y, z, \gamma) \right\}, \quad a \in \mathbb{S}_d^{>0}(\mathbb{R});$$

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \quad \text{and} \quad \hat{F}_t^0 := \hat{F}_t(0, 0)$$

$\mathcal{P}_H$ : set of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that

$$\underline{a}_{\mathbb{P}} \le \hat{a} \le \overline{a}_{\mathbb{P}}, \text{for some } \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \text{ and } \mathbb{E}^{\mathbb{P}}\left[\int_0^1 |\hat{F}_t^0|^2 dt\right] < \infty.$$

**Def** $\mathcal{P}_H-$q.s. means $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_H$ (Denis-Martini 04)

# Nonlinear generators

$H_t(\omega, y, z, \gamma) : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$
$D_H \subset \mathbb{R}^{d\times d}$ given, containing 0

- For fixed $(y, z, \gamma)$, $H$ is $\mathbb{F}-$progressively measurable
- $H$ is uniformly Lipschitz in $(y, z)$, lsc in $\gamma$
- $H$ is uniformly continuous in $\omega$ under the $\mathbb{L}^\infty-$norm

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} a : \gamma - H_t(\omega, y, z, \gamma) \right\}, \quad a \in \mathbb{S}_d^{>0}(\mathbb{R});$$

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \quad \text{and} \quad \hat{F}_t^0 := \hat{F}_t(0, 0)$$

$\mathcal{P}_H$ : set of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that

$$\underline{a}_{\mathbb{P}} \le \hat{a} \le \overline{a}_{\mathbb{P}}, \text{for some } \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \text{ and } \mathbb{E}^{\mathbb{P}}\left[ \int_0^1 |\hat{F}_t^0|^2 dt \right] < \infty.$$

**Def** $\mathcal{P}_H-$q.s. means $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_H$ (Denis-Martini 04)

# Nonlinear generators

$H_t(\omega, y, z, \gamma) : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$
$D_H \subset \mathbb{R}^{d\times d}$ given, containing 0

- For fixed $(y, z, \gamma)$, $H$ is $\mathbb{F}-$progressively measurable
- $H$ is uniformly Lipschitz in $(y, z)$, lsc in $\gamma$
- $H$ is uniformly continuous in $\omega$ under the $\mathbb{L}^\infty-$norm

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} a : \gamma - H_t(\omega, y, z, \gamma) \right\}, \quad a \in \mathbb{S}_d^{>0}(\mathbb{R});$$

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \quad \text{and} \quad \hat{F}_t^0 := \hat{F}_t(0, 0)$$

$\mathcal{P}_H$ : set of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that

$$\underline{a}_{\mathbb{P}} \le \hat{a} \le \overline{a}_{\mathbb{P}}, \text{for some } \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \text{ and } \mathbb{E}^{\mathbb{P}}\left[ \int_0^1 |\hat{F}_t^0|^2 dt \right] < \infty.$$

**Def** $\mathcal{P}_H-$q.s. means $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_H$ (Denis-Martini 04)

# Target problem and relaxations $\mathcal{V}(\xi) \geq \bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi)$

- $Z \in \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathcal{SM}^2(\mathbb{P})$, $d\langle Z, B\rangle_t = \Gamma_t d\langle B\rangle_t$, $\mathcal{P}_H-$q.s. and

$$\begin{aligned} Y_t^Z &= Y_0^Z - \int_0^t H_s(Y_s, Z_s, \Gamma_s)ds + \int_0^t Z_s \circ dB_s \\ \mathcal{V}(\xi) &:= \inf\left\{ Y_0 : \ Y_T^Z \geq \xi \ \mathcal{P}_H - \text{q.s. } Z \in \cap_{\mathbb{P}\in\mathcal{P}_H}\mathcal{SM}^2(\mathbb{P}) \right\} \end{aligned}$$

Relaxation 1

$$\bar{Y}_t^{\mathbb{P},\bar{Z},\bar{\Gamma}} = \bar{Y}_0 + \int_0^t \left( \tfrac{1}{2}\hat{a}_s : \bar{\Gamma}_s - H_s(\bar{Y}_s^{\mathbb{P},\bar{Z},\bar{\Gamma}}, \bar{Z}_s, \bar{\Gamma}_s) \right) ds + \int_0^t \bar{Z}_s dB_s,$$

$$\bar{\mathcal{V}}(\xi) := \inf\left\{ \bar{Y}_0 : \ \exists\ \bar{Z}, \bar{G} \in \cap_{\mathbb{P}\in\mathcal{P}_H}\mathbb{H}^2(\mathbb{P}), \ \ \bar{Y}_T^{\mathbb{P},\bar{Z}} \geq \xi \ \mathbb{P} - \text{a.s. } \mathbb{P}\in\mathcal{P}_H \right\}$$

Relaxation 2 $\bar{\bar{Y}}_t^{\mathbb{P},\bar{\bar{Z}}} = \bar{\bar{Y}}_0 + \int_0^t \hat{F}_s\left(\bar{\bar{Y}}_s, \bar{\bar{Z}}_s\right) ds + \int_0^t \bar{\bar{Z}}_s dB_s,$

$$\bar{\bar{\mathcal{V}}}(\xi) := \inf\left\{ \bar{\bar{Y}}_0 : \ \exists\ \bar{\bar{Z}} \in \cap_{\mathbb{P}\in\mathcal{P}_H}\mathbb{H}^2(\mathbb{P}), \ \ \bar{\bar{Y}}_T^{\mathbb{P},\bar{\bar{Z}}} \geq \xi \ \mathbb{P} - \text{a.s. } \mathbb{P}\in\mathcal{P}_H \right\}$$

# Target problem and relaxations $\mathcal{V}(\xi) \geq \bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi)$

- $Z \in \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathcal{SM}^2(\mathbb{P})$, $d\langle Z, B\rangle_t = \Gamma_t d\langle B\rangle_t$, $\mathcal{P}_H-$q.s. and

$$\begin{aligned} Y_t^Z &= Y_0^Z - \int_0^t H_s(Y_s, Z_s, \Gamma_s)ds + \int_0^t Z_s \circ dB_s \\ \mathcal{V}(\xi) &:= \inf\left\{Y_0 : \ Y_T^Z \geq \xi \ \mathcal{P}_H - \text{q.s. } Z \in \cap_{\mathbb{P}\in\mathcal{P}_H}\mathcal{SM}^2(\mathbb{P})\right\} \end{aligned}$$

**Relaxation 1**

$$\bar{Y}_t^{\mathbb{P},\bar{Z},\bar{\Gamma}} = \bar{Y}_0 + \int_0^t \left(\tfrac{1}{2}\hat{a}_s : \bar{\Gamma}_s - H_s(\bar{Y}_s^{\mathbb{P},\bar{Z},\bar{\Gamma}}, \bar{Z}_s, \bar{\Gamma}_s)\right) ds + \int_0^t \bar{Z}_s dB_s,$$

$$\bar{\mathcal{V}}(\xi) := \inf\left\{\bar{Y}_0 : \ \exists\ \bar{Z}, \bar{G} \in \cap_{\mathbb{P}\in\mathcal{P}_H}\mathbb{H}^2(\mathbb{P}), \ \bar{Y}_T^{\mathbb{P},\bar{Z}} \geq \xi \ \mathbb{P} - \text{a.s. } \mathbb{P}\in\mathcal{P}_H\right\}$$

**Relaxation 2** $\bar{\bar{Y}}_t^{\mathbb{P},\bar{\bar{Z}}} = \bar{\bar{Y}}_0 + \int_0^t \hat{F}_s\left(\bar{\bar{Y}}_s, \bar{\bar{Z}}_s\right) ds + \int_0^t \bar{\bar{Z}}_s dB_s,$

$$\bar{\bar{\mathcal{V}}}(\xi) := \inf\left\{\bar{\bar{Y}}_0 : \ \exists\ \bar{\bar{Z}} \in \cap_{\mathbb{P}\in\mathcal{P}_H}\mathbb{H}^2(\mathbb{P}), \ \bar{\bar{Y}}_T^{\mathbb{P},\bar{\bar{Z}}} \geq \xi \ \mathbb{P} - \text{a.s. } \mathbb{P}\in\mathcal{P}_H\right\}$$

# Target problem and relaxations $\mathcal{V}(\xi) \geq \bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi)$

- $Z \in \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathcal{SM}^2(\mathbb{P})$, $d\langle Z, B\rangle_t = \Gamma_t d\langle B\rangle_t$, $\mathcal{P}_H-$q.s. and

$$
\begin{aligned}
Y_t^Z &= Y_0^Z - \int_0^t H_s(Y_s, Z_s, \Gamma_s)ds + \int_0^t Z_s \circ dB_s \\
\mathcal{V}(\xi) &:= \inf\left\{Y_0 : \; Y_T^Z \geq \xi \; \mathcal{P}_H - \text{q.s. } Z \in \cap_{\mathbb{P}\in\mathcal{P}_H}\mathcal{SM}^2(\mathbb{P})\right\}
\end{aligned}
$$

**Relaxation 1**

$$\bar{Y}_t^{\mathbb{P},\bar{Z},\bar{\Gamma}} = \bar{Y}_0 + \int_0^t \left(\tfrac{1}{2}\hat{a}_s : \bar{\Gamma}_s - H_s(\bar{Y}_s^{\mathbb{P},\bar{Z},\bar{\Gamma}}, \bar{Z}_s, \bar{\Gamma}_s)\right) ds + \int_0^t \bar{Z}_s dB_s,$$

$$\bar{\mathcal{V}}(\xi) := \inf\left\{\bar{Y}_0 : \; \exists\, \bar{Z}, \bar{G} \in \cap_{\mathbb{P}\in\mathcal{P}_H}\mathbb{H}^2(\mathbb{P}), \; \bar{Y}_T^{\mathbb{P},\bar{Z}} \geq \xi \; \mathbb{P} - \text{a.s. } \mathbb{P} \in \mathcal{P}_H\right\}$$

**Relaxation 2** $\bar{\bar{Y}}_t^{\mathbb{P},\bar{\bar{Z}}} = \bar{\bar{Y}}_0 + \int_0^t \hat{F}_s\left(\bar{\bar{Y}}_s, \bar{\bar{Z}}_s\right) ds + \int_0^t \bar{\bar{Z}}_s dB_s,$

$$\bar{\bar{\mathcal{V}}}(\xi) := \inf\left\{\bar{\bar{Y}}_0 : \; \exists\, \bar{\bar{Z}} \in \cap_{\mathbb{P}\in\mathcal{P}_H}\mathbb{H}^2(\mathbb{P}), \; \bar{\bar{Y}}_T^{\mathbb{P},\bar{\bar{Z}}} \geq \xi \; \mathbb{P} - \text{a.s. } \mathbb{P} \in \mathcal{P}_H\right\}$$

# Definition

For $\mathcal{F}_1-$meas. $\xi$, consider the 2BSDE :

$$dY_t = \hat{F}_t(Y_t, Z_t)dt + Z_t dB_t - dK_t, \; 0 \le t \le 1, \; Y_T = \xi, \; \mathcal{P}_H - \text{q.s.}$$

We say $(Y, Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ is a solution to the 2BSDE if

- $Y_T = \xi$, $\mathcal{P}_H-$q.s.
- For each $\mathbb{P} \in \mathcal{P}_H$, $K^{\mathbb{P}}$ has nondecreasing paths, $\mathbb{P}-$a.s. :

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s, \;\; t \in [0,1], \;\; \mathbb{P} - \text{a.s.}$$

- The family of processes $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$ satisfies :

$$K_t^{\mathbb{P}} = \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[K_1^{\mathbb{P}'}], \;\; \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H, t \le T$$

# Definition

For $\mathcal{F}_1-$meas. $\xi$, consider the 2BSDE :

$$dY_t = \hat{F}_t(Y_t, Z_t)dt + Z_t dB_t - dK_t, \; 0 \le t \le 1, \; Y_T = \xi, \; \mathcal{P}_H - \text{q.s.}$$

We say $(Y, Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ is a solution to the 2BSDE if

- $Y_T = \xi$, $\mathcal{P}_H-$q.s.
- For each $\mathbb{P} \in \mathcal{P}_H$, $K^{\mathbb{P}}$ has nondecreasing paths, $\mathbb{P}-$a.s. :

$$K^{\mathbb{P}}_t := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s, \; t \in [0,1], \; \mathbb{P} - \text{a.s.}$$

- The family of processes $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$ satisfies :

$$K^{\mathbb{P}}_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})}{\text{ess inf}} \; \mathbb{E}^{\mathbb{P}'}_t[K^{\mathbb{P}'}_1], \; \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H, t \le T$$

# Definition

For $\mathcal{F}_1-$meas. $\xi$, consider the 2BSDE :

$$dY_t = \hat{F}_t(Y_t, Z_t)dt + Z_t dB_t - dK_t, \; 0 \le t \le 1, \; Y_T = \xi, \; \mathcal{P}_H - \text{q.s.}$$

We say $(Y, Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ is a solution to the 2BSDE if

- $Y_T = \xi$, $\mathcal{P}_H-$q.s.
- For each $\mathbb{P} \in \mathcal{P}_H$, $K^{\mathbb{P}}$ has nondecreasing paths, $\mathbb{P}-$a.s. :

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s, \; t \in [0,1], \; \mathbb{P} - \text{a.s.}$$

- The family of processes $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$ satisfies :

$$K_t^{\mathbb{P}} = \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[K_1^{\mathbb{P}'}], \quad \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H, t \le T$$

# Definition

For $\mathcal{F}_1-$meas. $\xi$, consider the 2BSDE :

$$dY_t = \hat{F}_t(Y_t, Z_t)dt + Z_t dB_t - dK_t,\ 0 \le t \le 1,\ \ Y_T = \xi,\ \mathcal{P}_H - \text{q.s.}$$

We say $(Y, Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ is a solution to the 2BSDE if

- $Y_T = \xi$, $\mathcal{P}_H-$q.s.
- For each $\mathbb{P} \in \mathcal{P}_H$, $K^{\mathbb{P}}$ has nondecreasing paths, $\mathbb{P}-$a.s. :

$$K^{\mathbb{P}}_t := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ \ t \in [0,1],\ \ \mathbb{P} - \text{a.s.}$$

- The family of processes $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$ satisfies :

$$K^{\mathbb{P}}_t = \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_t[K^{\mathbb{P}'}_1],\ \ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H, t \le T$$

# Back to standard BSDEs

For standard BSDEs

$$dY_t := -H_t^0(Y_t, Z_t)dt + Z_t dB_t, \;\; Y_T = \xi$$

the nonlinearity and the corresponding conjugate :

$$H_t(., \gamma) := \frac{-1}{2} I_d : \gamma + H_t^0(.), \quad F_t(., a) = \begin{cases} -H_t^0(.) & \text{for } a = I_d \\ \infty & \text{otherwise} \end{cases}$$

Then $\mathcal{P} = \{\mathbb{P}^0\}$, $K^{\mathbb{P}^0} \equiv 0$, and the previous definition reduces to the standard definition

$$dY_t := -H_t^0(Y_t, Z_t)dt + Z_t dB_t, \;\; Y_T = \xi, \;\; \mathbb{P}^0 - \text{a.s.}$$

# Benchmark example : uncertain volatility model, $G$−expectation (Peng)

Let $d = 1$, and $H_t(y, z, \gamma) := G(\gamma) = \bar{a}\gamma^+ - \underline{a}\gamma^-$, and suppose that the PDE

$$\frac{\partial u}{\partial t} + G(u_{xx}) = 0, \quad \text{and} \quad u(T, .) = g$$

has a smooth solution. Then

$$Y_t := u(t, B_t), \quad Z_t := Du(t, B_t),$$

is a solution of the 2BSDE with

$$K_t \quad := \quad \int_0^t \left( G(u_{xx}) - \frac{1}{2}\hat{a}_s u_{xx} \right)(s, B_s)ds$$

# Outline

# Outline

# Spaces and norms

- $\mathbb{L}^p_H := \{\xi\ \mathcal{F}_1 - \text{meas.}: \ \|\xi\|^p_{\mathbb{L}^p_H} < \infty\}$
- $\mathbb{H}^p_H := \{Z\ \mathbb{F}^+\text{–prog. meas. in } \mathbb{R}^d : \ \|Z\|^p_{\mathbb{H}^p_H} < \infty\}$

$$\|\xi\|^p_{\mathbb{L}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[|\xi|^p], \ \|Z\|^p_{\mathbb{H}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt\Big)^{p/2}\Big]$$

- $\mathbb{D}^p_H := \{Y\ \mathbb{F}^+\text{–prog. in } \mathbb{R} \text{ càdlàg } \mathcal{P}_H - \text{q.s.}\ \|Y\|^p_{\mathbb{D}^p_H} < \infty\}$

$$\|Y\|^p_{\mathbb{D}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} |Y_t|^p\Big]$$

- $\mathbb{L}^2_{H,*} := \{\xi \in \mathbb{L}^2_H : \|\xi\|_{\mathbb{L}^2_{H,*}} < \infty\}$

$$\|\xi\|_{\mathbb{L}^2_{H,*}} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} \mathbb{E}^{H,\mathbb{P}}_t[|\xi|^2]\Big], \ \mathbb{E}^{H,\mathbb{P}}_t[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t]$$

- $\hat{\mathbb{L}}^2_H :=$ closure of $\mathrm{UC}_b(\Omega)$ under the norm $\|\cdot\|_{\mathbb{L}^2_{H,*}}$

# Spaces and norms

- $\mathbb{L}^p_H := \left\{ \xi \ \mathcal{F}_1 - \text{meas.} : \ \|\xi\|^p_{\mathbb{L}^p_H} < \infty \right\}$
- $\mathbb{H}^p_H := \left\{ Z \ \mathbb{F}^+ - \text{prog. meas. in } \mathbb{R}^d : \ \|Z\|^p_{\mathbb{H}^p_H} < \infty \right\}$

$$\|\xi\|^p_{\mathbb{L}^p_H} := \sup_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[|\xi|^p], \ \|Z\|^p_{\mathbb{H}^p_H} := \sup_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[ \left( \int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt \right)^{p/2} \right]$$

- $\mathbb{D}^p_H := \left\{ Y \ \mathbb{F}^+ - \text{prog. in } \mathbb{R} \text{ càdlàg } \mathcal{P}_H - \text{q.s. } \|Y\|^p_{\mathbb{D}^p_H} < \infty \right\}$

$$\|Y\|^p_{\mathbb{D}^p_H} := \sup_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[ \sup_{0 \le t \le 1} |Y_t|^p \right]$$

- $\mathbb{L}^2_{H,*} := \left\{ \xi \in \mathbb{L}^2_H : \|\xi\|_{\mathbb{L}^2_{H,*}} < \infty \right\}$

$$\|\xi\|_{\mathbb{L}^2_{H,*}} := \sup_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[ \sup_{0 \le t \le 1} \mathbb{E}_t^{H,\mathbb{P}}[|\xi|^2] \right], \ \mathbb{E}_t^{H,\mathbb{P}}[\xi] := \underset{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})}{\text{ess sup}}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'}[\xi | \mathcal{F}_t]$$

- $\hat{\mathbb{L}}^2_H :=$ closure of $\mathrm{UC}_b(\Omega)$ under the norm $\|\cdot\|_{\mathbb{L}^2_{H,*}}$

# Spaces and norms

- $\mathbb{L}^p_H := \left\{\xi \ \mathcal{F}_1 - \text{meas.} : \ \|\xi\|^p_{\mathbb{L}^p_H} < \infty\right\}$
- $\mathbb{H}^p_H := \left\{Z \ \mathbb{F}^+ - \text{prog. meas. in } \mathbb{R}^d : \ \|Z\|^p_{\mathbb{H}^p_H} < \infty\right\}$

$$\|\xi\|^p_{\mathbb{L}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[|\xi|^p], \ \|Z\|^p_{\mathbb{H}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt\right)^{p/2}\right]$$

- $\mathbb{D}^p_H := \left\{Y \ \mathbb{F}^+ - \text{prog. in } \mathbb{R} \text{ càdlàg } \mathcal{P}_H - \text{q.s. } \|Y\|^p_{\mathbb{D}^p_H} < \infty\right\}$

$$\|Y\|^p_{\mathbb{D}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} |Y_t|^p\right]$$

- $\mathbb{L}^2_{H,*} := \left\{\xi \in \mathbb{L}^2_H : \|\xi\|_{\mathbb{L}^2_{H,*}} < \infty\right\}$

$$\|\xi\|_{\mathbb{L}^2_{H,*}} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} \mathbb{E}_t^{H,\mathbb{P}}[|\xi|^2]\right], \ \mathbb{E}_t^{H,\mathbb{P}}[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t]$$

- $\hat{\mathbb{L}}^2_H :=$ closure of $\mathrm{UC}_b(\Omega)$ under the norm $\|\cdot\|_{\mathbb{L}^2_{H,*}}$

# Spaces and norms

- $\mathbb{L}^p_H := \{\xi\ \mathcal{F}_1 - \text{meas.} :\ \|\xi\|^p_{\mathbb{L}^p_H} < \infty\}$
- $\mathbb{H}^p_H := \{Z\ \mathbb{F}^+ - \text{prog. meas. in } \mathbb{R}^d :\ \|Z\|^p_{\mathbb{H}^p_H} < \infty\}$

$$\|\xi\|^p_{\mathbb{L}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[|\xi|^p],\ \|Z\|^p_{\mathbb{H}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt\Big)^{p/2}\Big]$$

- $\mathbb{D}^p_H := \{Y\ \mathbb{F}^+ - \text{prog. in } \mathbb{R} \text{ càdlàg } \mathcal{P}_H - \text{q.s. } \|Y\|^p_{\mathbb{D}^p_H} < \infty\}$

$$\|Y\|^p_{\mathbb{D}^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} |Y_t|^p\Big]$$

- $\mathbb{L}^2_{H,*} := \{\xi \in \mathbb{L}^2_H : \|\xi\|_{\mathbb{L}^2_{H,*}} < \infty\}$

$$\|\xi\|_{\mathbb{L}^2_{H,*}} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} \mathbb{E}_t^{H,\mathbb{P}}[|\xi|^2]\Big],\ \mathbb{E}_t^{H,\mathbb{P}}[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t]$$

- $\hat{\mathbb{L}}^2_H :=$ closure of $\mathrm{UC}_b(\Omega)$ under the norm $\|\cdot\|_{\mathbb{L}^2_{H,*}}$

# Representation and uniqueness

**Assumption** $\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} \mathbb{E}_t^{H,\mathbb{P}}\big[\int_t^1 |\hat{F}_s^0|^2 ds\big]\Big] < \infty$, and :

$$\big|\hat{F}_t(y,z_1) - \hat{F}_t(y,z_2)\big| \le C|\hat{a}_t^{1/2}(z_1 - z_2)|, \ dt \times d\mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H$$

**Theorem** Let $\xi \in \mathbb{L}^2_{H,*}$ and suppose $(Y,Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ is a solution of the 2BSDE. Then, for any $\mathbb{P} \in \mathcal{P}_H$ and $0 \le t_1 < t_2 \le 1$,

$$Y_{t_1} = \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H(t_1,\mathbb{P})}{}^{\mathbb{P}} \ \mathcal{Y}^{\mathbb{P}'}_{t_1}(t_2, Y_{t_2}), \quad \mathbb{P} - \text{a.s.}$$

$\mathcal{Y}^{\mathbb{P}}_{t_1}(t_2, Y_{t_2}) = y_{t_1}$, where $(y_t)_{t\le t_2}$ is the solution of

$$y_t = Y_{t_2} - \int_t^T \hat{F}_t(y_s, z_s)ds + \int_t^T z_s dB_s, \ \mathbb{P} - \text{a.s.}$$

**Corollary** The 2BSDE has at most one solution in $\mathbb{D}^2_H \times \mathbb{H}^2_H$, and comparison holds true

# Outline

# A priori estimates

**Theorem** (i) Let $\xi \in \mathbb{L}^2_{H,*}$ and $(Y,Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ a solution of the 2BSDE. Then

$$\|Y\|^2_{\mathbb{D}^2_H} + \|Z\|^2_{\mathbb{H}^2_H} + \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[|K_1^{\mathbb{P}}|^2] \le C\left(\|\xi\|^2_{\mathbb{L}^2_{H,*}} + \|\hat{F}^0\|^2_{\mathbb{H}^2_{H,*}}\right)$$

(ii) Let $\xi^i \in \mathbb{L}^2_{H,*}$ and $(Y^i, Z^i) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ corresponding solutions to the 2BSDE, $i = 1,2$. Then, with $\delta\xi := \xi^1 - \xi^2$, $\delta Y := Y^1 - Y^2$, $\delta Z := Z^1 - Z^2$, and $\delta K^{\mathbb{P}} := K^{1,\mathbb{P}} - K^{2,\mathbb{P}}$ :

$$\begin{aligned}
\|\delta Y\|_{\mathbb{D}^2_H} &\le C\|\delta\xi\|_{\mathbb{L}^2_{H,*}} \quad \text{and} \\
\|\delta Z\|^2_{\mathbb{H}^2_H} &+ \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} |\delta K_t^{\mathbb{P}}|^2\right] \\
&\le C\|\delta\xi\|^2_{\mathbb{L}^2_{H,*}} + C\left(\|\xi^1\|_{\mathbb{L}^2_{H,*}} + \|\hat{F}^0\|_{\mathbb{H}^2_{H,*}}\right)\|\delta\xi\|_{\mathbb{L}^2_{H,*}}
\end{aligned}$$

# A priori estimates

**Theorem** (i) Let $\xi \in \mathbb{L}^2_{H,*}$ and $(Y,Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ a solution of the 2BSDE. Then

$$\|Y\|^2_{\mathbb{D}^2_H} + \|Z\|^2_{\mathbb{H}^2_H} + \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[|K_1^{\mathbb{P}}|^2] \le C\big(\|\xi\|^2_{\mathbb{L}^2_{H,*}} + \|\hat{F}^0\|^2_{\mathbb{H}^2_{H,*}}\big)$$

(ii) Let $\xi^i \in \mathbb{L}^2_{H,*}$ and $(Y^i, Z^i) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ corresponding solutions to the 2BSDE, $i=1,2$. Then, with $\delta\xi := \xi^1 - \xi^2$, $\delta Y := Y^1 - Y^2$, $\delta Z := Z^1 - Z^2$, and $\delta K^{\mathbb{P}} := K^{1,\mathbb{P}} - K^{2,\mathbb{P}}$ :

$$\begin{aligned}
\|\delta Y\|_{\mathbb{D}^2_H} &\le C\|\delta\xi\|_{\mathbb{L}^2_{H,*}} \quad \text{and} \\
\|\delta Z\|^2_{\mathbb{H}^2_H} &+ \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} |\delta K_t^{\mathbb{P}}|^2\Big] \\
&\le C\|\delta\xi\|^2_{\mathbb{L}^2_{H,*}} + C\big(\|\xi^1\|_{\mathbb{L}^2_{H,*}} + \|\hat{F}^0\|_{\mathbb{H}^2_{H,*}}\big)\|\delta\xi\|_{\mathbb{L}^2_{H,*}}
\end{aligned}$$

# Existence

**Theorem** For any $\xi \in \hat{\mathbb{L}}^2_H$, the 2BSDE admits a unique solution $(Y,Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$.

Recall $\hat{\mathbb{L}}^2_H :=$ closure of $\mathrm{UC}_b(\Omega)$ under the norm $\|\cdot\|_{\mathbb{L}^2_{H,*}}$, where

$$\|\xi\|_{\mathbb{L}^2_{H,*}} \quad := \quad \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} \mathbb{E}^{H,\mathbb{P}}_t[|\xi|^2]\Big]$$

and

$$\mathbb{E}^{H,\mathbb{P}}_t[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t]$$

# Outline

1. Motivations
   - Hedging under Gamma constraints
   - Hedging under market illiquidity
   - Probabilistic numerics for nonlinear PDEs
2. Backward SDEs : a quick review
3. 2nd order backward SDEs
   - The CSTV framework
   - An alternative formulation of 2BSDEs
4. Main results
   - Representation, uniqueness
   - Existence
   - Connection with PDEs and stochastic control

# Connection with PDEs

**Theorem** Under "natural conditions", the solution of the 2BSDE satisfies $Y_t = u(t, B_t)$, $t \in [0, T]$, $\mathcal{P}_H-$q.s. and $u$ is a viscosity solution of

$$\frac{\partial u}{\partial t}(t,x) + \hat{H}\Big(t,x,u(t,x),Du(t,x),D^2u(t,x)\Big) \quad = \quad 0, \; 0 \le t < 1$$
$$u(1,x) = g(x)$$

where

$$\hat{H}(t,x,y,z,\gamma) \quad = \quad \sup_{a \in \mathbb{S}_d^+(\mathbb{R})} \left\{\frac{1}{2}a : \gamma - F(t,x,y,z,a)\right\}, \; \gamma \in \mathbb{R}^{d\times d}.$$

We also have a Feynman-Kac representation theorem for the Cauchy problem with the latter fully nonlinear PDE

# Connection with $G-$expectation

Denis-Martini (2004), Peng's $G-$expectation (2007) :

$$H_t(y,z,\gamma) = G(\gamma) := \frac{1}{2} \sup_{\underline{a} \le a \le \overline{a}} a:\gamma \quad \text{Then} \quad F_t(a) = \begin{cases} 0 & \text{on } [\underline{a}, \overline{a}] \\ \infty & \text{otherwise} \end{cases}$$

The corresponding PDE is

$$\frac{\partial u}{\partial t} + G\left(D^2 u\right) \quad = \quad 0$$

given the terminal data $u(T,.) = g$, this is the DPE for the problem of superhedging under uncertain volatility :

$$\inf\left\{ Y_0 : \ Y_0 + \int_0^T Z_s dB_s \ge g(B_T), \ \mathcal{P}_H - \text{q.s.} \right\}$$

where $\mathcal{P}_H = \left\{ \mathbb{P} \in \mathcal{P}_S : \ \hat{a} \in [\underline{a}, \overline{a}], \mathbb{P} - \text{a.s.} \right\}$

# Connection with stochastic control

In classical stochastic control theory, define :

$$Y_t^{\mathbb{P}} \quad := \quad \underset{\mathbb{P} \in \mathcal{P}_H(t,\mathbb{P})}{\operatorname{ess\,sup}} \ \mathbb{E}\left[\xi | \mathcal{F}_t\right], \quad \mathbb{P}-\text{a.s. for all } \mathbb{P} \in \mathcal{P}_H$$

Then

- $\{Y_t^{\mathbb{P}}\}$ can be aggregated into a $\mathbb{P}-$supermartingale for all $\mathbb{P} \in \mathcal{P}_H$
- $\{Y_t^{\mathbb{P}^*}\}$ is a $\mathbb{P}^*-$martingale for some $\mathbb{P}^* \in \mathcal{P}_H \Longrightarrow \mathbb{P}^*$ optimal

However, it is not clear how to aggregate the family of processes

$$\left\{Y^{\mathbb{P}}, \ \mathbb{P} \in \mathcal{P}_H\right\}$$

i.e. find a process $Y$ such that $Y = Y^{\mathbb{P}}$, $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_H$