Optimal investment on finite horizon with random discrete order flow in illiquid markets

Mihai Sîrbu, University of Texas at Austin

based on joint work with
Paul Gassiat and Huyên Pham from University Paris 7

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Outline

Objective

build and analyze a model of optimal investment which accounts for

- time illiquidity: the impossibility to trade at all times
- more frequent trading near the finite time horizon

Modeling Time Illiquidity

- finite horizon $T < \infty$
- money market paying zero risk-free interest rate
- a risky asset S traded/observed *only* at some some exogenous random times $(\tau_n)_{n \geq 0}$

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < T.$$

What does an agent actually observe up to time $t \leq T$?

$$(\tau_0, S_{\tau_0}), (\tau_1, S_{\tau_1}), \ldots (\tau_n, S_{\tau_n}),$$

if

$$\tau_n \leq t < \tau_{n+1}.$$

More Structure for the Model

- the discrete-time observed asset prices $(S_{\tau_n})_{n\geq 0}$ come from an *unobserved* continuous-time stochastic process $(S_t)_{0\leq t\leq T}$ (based on fundamentals)
- S is a stochastic exponential

$$S = \mathscr{E}(L),$$

  where L is a time inhomogeneous Lévy process

$$L_t = \int_0^t b(u)\, du + \int_0^t c(u)\, dB_u + \int_0^t \int_{-1}^{\infty} y(\mu(dt, dy) - \nu(dt, dy))$$

  with $\Delta L > -1$
- $(\tau_n)_{n\geq 0}$ and $(S_t)_{0\leq t\leq T}$ are independent under the physical probability measure $\mathbb{P}$.

More Structure for the Model cont'd

Denote by $Z_{t,s}$ the *unobserved return* between the times $t \leq s$

$$Z_{t,s} = \frac{S_s - S_t}{S_t},$$

and

$$p(t, s, dz) = \mathbb{P}[Z_{t,s} \in dz],$$

the distribution of the return.

Remark: based on the assumptions on L, $p(t, s, dz)$ has full support on $(-1, \infty)$.

Observation/Trading Times

Recall the sequence of exogenous random times $(\tau_n)_{n\geq 0}$ when observation/trading takes place.
Need

- to be able to model *more frequent trading* near the horizon T
- to obtain a reasonable mathematical structure

Solution: assume that $(\tau_n)_{n\geq 0}$ are the jump times of an inhomogeneous Poisson process with deterministic intensity.

Time Inhomogeneous Poisson Processes

Consider a (deterministic) intensity $t \in [0, T) \to \lambda(t) \in (0, \infty)$, such that:

$$\int_0^t \lambda(u)du \; < \; \infty, \; (\forall) \; 0 \leq t < T \text{ and } \int_0^T \lambda(u)du \; = \; \infty.$$

Define

- $N_t = M_{\int_0^t \lambda(s)ds} \quad 0 \leq t < T$, where M is a Poisson process with intensity 1.
- $(\tau_n)_{n \geq 0}$ as the sequence of jumps of N.

Consequences:

- we have an increasing sequence of times that accumulates at T
- $$\mathbb{P}[\tau_{n+1} \in ds | \tau_n = t] = \lambda(s) e^{-\int_t^s \lambda(u)du} 1_{\{t \leq s < T\}} \, ds$$

Trading Strategies

At any of the exogenous trading times τ_{n-1} the agent can choose to hold α_n units of the risky asset up to the next trading time τ_n
What information is available in order to choose α_n? Define the discrete filtration:

$$\mathscr{F}_n = \sigma\Big\{(\tau_k, S_{\tau_k}) : 1 \leq k \leq n\Big\} = \sigma\Big\{(\tau_k, Z_k) : 1 \leq k \leq n\Big\},\ n \geq 1$$

where

$$Z_n = Z_{\tau_{n-1},\tau_n}, \qquad n \geq 1,$$

is the observed return.
In this model a trading strategy is a real-valued $\mathscr{F}$-predictable process $\alpha = (\alpha_n)_{n\geq 1}$, where α_n represents the amount invested in the stock over the period $(\tau_{n-1}, \tau_n]$ after observing the stock price at time τ_{n-1}

$$\alpha_n \in \mathscr{F}_{n-1}$$

Wealth Processes and Admissibility

Fix the initial wealth $X_0 > 0$.
Observed wealth process is defined by

$$X_{\tau_n} = X_{\tau_{n-1}} + \alpha_n Z_n, \quad n \geq 1.$$

Admissibility condition:

$$X_{\tau_n} \geq 0, \quad n \geq 1.$$

Denote by $\mathscr{A}$ the set of all admissible strategies.
Terminal wealth:

$$X_T \ = \ \lim_{n\to\infty} X_{\tau_n} = X_0 + \sum_{n=1}^{\infty} \alpha_n Z_n.$$

Does that limit exist?
Yes, if a martingale measure for S exists.

Distribution of the Observed Returns

the independence of S and the trading times ensures that for all n, the (regular) distribution of (τ_{n+1}, Z_{n+1}) conditioned on $\mathscr{F}_n$ is given as follows:

1. $\mathbb{P}[\tau_{n+1} \in ds | \mathscr{F}_n] = \lambda(s) e^{-\int_{\tau_n}^{s} \lambda(u) du} ds$
2. further conditioning on knowing the next arrival time τ_{n+1}, the return Z_{n+1} has distribution

$$\mathbb{P}[Z_{n+1} \in dz | \mathscr{F}_n \vee \sigma(\tau_{n+1})] = p(\tau_n, \tau_{n+1}, dz).$$

One consequence: Z_{n+1} has full support in $(-1, \infty)$.

More on Admissibility

Recall

$$X_{\tau_n} = X_{\tau_{n-1}} + \alpha_n Z_n \geq 0, \quad n \geq 1,$$

and Z_n has full support in $(-1, \infty)$, so admissibility means

$$0 \leq \alpha_n \leq X_{\tau_{n-1}}, \quad \text{for all } n \geq 1.$$

Since $Z_n > -1$ for each n, then $X_{\tau_n} > 0$ for each n.
We can use $X_{\tau_{n-1}} > 0$, to represent the trading strategy in terms of the *proportion* of the wealth invested in the risky asset at time τ_{n-1}

$$\pi_n = \alpha_n / X_{\tau_{n-1}},$$

as

$$0 \leq \pi_n \leq 1.$$

(Short sale constraints)

The Optimal Investment Problem

Find the strategy α which attains the supremum in

$$V_0 = \sup_{\alpha \in \mathscr{A}} \mathbb{E}[U(X_T)],$$

where the utility function U is defined on $(0, \infty)$

- strictly increasing
- strictly concave and C^1 on $(0, \infty)$
- satisfies the Inada conditions: $U'(0^+) = \infty$, $U'(\infty) = 0$.
  Note: actually a little more is needed, like power behavior close to 0 and ∞

(Direct) Dynamic Programming

Idea:

1. use the Markov structure of the problem to write (formally) the Dynamic Programming Equation
2. solve the equation analytically
3. use "verification arguments" to show that the solution found above is the value function, and find the optimal strategy in "feedback form"

1-Dynamic Programming Equation (DPE)

The control problem is

- finite horizon in time t
- infinite horizon with respect to the number of trades/observations n

Look for a function $v(t, x)$ such that

- for each $\alpha \in \mathscr{A}$ we have that $\{v(\tau_n, X_{\tau_n}), n \geq 0\}$ is a $(\mathbb{P}, \mathscr{F})$-supermartingale
- for *some* $\alpha^* \in \mathscr{A}$ we have that $\{v(\tau_n, X^*_{\tau_n}), n \geq 0\}$ is a $(\mathbb{P}, \mathscr{F})$-martingale
- $\lim_{t \to T, y \to x} v(t, y) = U(x)$

More on DPE

Because of the (conditional) distribution of observed returns, we have

$$\begin{aligned}
&\mathbb{E}\big[v(\tau_{n+1}, X_{\tau_{n+1}})|\mathscr{F}_n\big] = \\
= \; & \int_{\tau_n}^{T} \int_{(-1,\infty)} \lambda(s) e^{-\int_{\tau_n}^{s} \lambda(u)du} v(s, X_{\tau_n} + \alpha_{n+1} z) p(\tau_n, s, dz) ds \\
\geq \; & (= \text{if optimal}) v(\tau_n, X_{\tau_n})
\end{aligned}$$

so that $\{v(\tau_n, X_{\tau_n}), n \geq 0\}$ is a supermartingale or martingale.

DPE cont'd

We have the equation

$$v(t,x) = \sup_{a\in[0,x]} \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} v(s, x+az) p(t,s,dz) ds,$$
$$= \sup_{\pi\in[0,1]} \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} v(s, x(1+\pi z)) p(t,s,dz) ds,$$

for all $(t,x) \in [0,T) \times (0,\infty)$ together with the terminal condition

$$\lim_{t\nearrow T, x'\to x} v(t,x') = U(x), \quad x > 0.$$

2-Solving the DPE

Denote by

$$\mathscr{L} v(t,x) = \sup_{a\in[0,x]} \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} v(s, x+az) p(t,s,dz) ds.$$

Can rewrite the (DPE) as

$$\begin{cases} \mathscr{L} v = v \\ \lim_{t \nearrow T, x' \to x} v(t,x') = U(x). \end{cases}$$

How do we find a solution: by monotone iterations.

$$v_0(t,x) = U(x), \quad v_{n+1} = \mathscr{L} v_n$$

We have $v_0 \leq v_1 \leq \cdots \leq v_n$ and $v_n \nearrow v$, where v is a solution of the (DPE).

3-Verification

Fix $\alpha \in \mathscr{A}$. We have

$$v(0, X_0) \geq \mathbb{E}[v(\tau_n, X_{\tau_n})]$$

IF we have some uniform integrability conditions (which we do check!), together with $\lim_{t \to T, y \to x} v(t, y) = U(x)$ we get

$$v(0, X_0) \geq \mathbb{E}[U(X_T)], \quad (\forall)\ \alpha \in \mathscr{A}.$$

3-Verification, the Optimal Strategy

Denote by $\alpha^*(t,x)$ the argmax in the DPE. For the feedback control

$$\alpha_{n+1} = \alpha^*(\tau_n, X_{\tau_n}), \quad n \geq 0,$$

the state equation has to be solved recursively to obtain the wealth process $(X^*_{\tau_n})_{n\geq 0}$ (and the control α^*).
From the DPE we have that

$$\{v(\tau_n, X^*_{\tau_n}), n \geq 0\}$$

is a $(\mathbb{P}, \mathscr{F})$-martingale so

$$v(0, X_0) = \mathbb{E}[v(\tau_n, X^*_{\tau_n})].$$

Need uniform integrability again to pass to the limit and get

$$v(0, X_0) = \mathbb{E}[U(X^*_T)]$$

Conclusions:

- $V_0 = v(0, X_0)$
- the feedback α^* which makes $\{v(\tau_n, X^*_{\tau_n}), n \geq 0\}$ a $(\mathbb{P}, \mathscr{F})$-martingale is optimal

Some Technical Details

- nee to show that

$$v := \sup_n v_n < \infty$$

without using the dynamic programming principle

- we need controls on the jump measure, compatible with the utility function to get the uniform integrability

Uniform Integrability

Assumptions on the utility function:
(i) there exist some constants $C > 0$ and $p \in (0,1)$ such that

$$U^+(x) \leq C(1 + x^p), \quad (\forall)\ x > 0$$

(ii) Either $U(0) > -\infty$, or $U(0) = -\infty$ and there exist some constants $C' > 0$ and $p' < 0$ such that

$$U^-(x) \leq C(1 + x^{p'}), \quad (\forall)\ x > 0.$$

Assumptions on the jump measure: (i) there exists $q > 1$ such that

$$\int_0^T \int_0^\infty \Big((1+y)^q - 1 - qy\Big)\nu(dt, dy) < \infty.$$

(ii) If the utility function U satisfies $U(0) = -\infty$, then there exists $r < p' < 0$ such that

$$\int_0^T \int_{-1}^0 \Big((1+y)^r - 1 - ry\Big)\nu(dt, dy) < \infty.$$

(iii) there are no predictable jumps, i.e. $\nu(\{t\}, (-1, \infty)) = 0$ for each t

The Approximate Solution v_n

Using again the same kind of verification arguments, we obtain as a by-product that

$$v_n(0, X_0) = \sup_{\alpha \in \mathscr{A}_n} \mathbb{E}[U(X_T)],$$

where $\mathscr{A}_n$ is the set of admissible controls $(\alpha_n)_{n \geq 1}$ such that

$$\alpha_{n+1} = \alpha_{n+2} = ... = 0.$$

CRRA Utility Functions

Power utility functions:

$$U(x) = \frac{x^\gamma}{\gamma}, \quad x > 0,\ \gamma < 1, \gamma \neq 0.$$

The value function has the form:

$$v(t,x) = \varphi(t)U(x).$$

The DPE becomes

$$\varphi(t) = \sup_{\pi\in[0,1]} \int_t^T \lambda(s) e^{-\int_t^s \lambda(u)du} \varphi(s) \Big(\int_{(-1,\infty)} (1+\pi z)^\gamma p(t,s,dz) \Big) ds$$

with terminal condition

$$\varphi(T) = 1.$$

Overview of the Solution

- derive (formally in the beginning) the DPE, having in mind the verification arguments
- solve the DPE analytically
- go over the verification arguments to compute the maximal expected utility and the optimal control in feedback form

Asymptotic Behavior

Question: what happens if intensity is very large at all times, not only close to maturity?
For a fixed intensity function λ, denote

$$V_0^\lambda = \sup_{\alpha \in \mathscr{A}^\lambda} \mathbb{E}[U(X_T)].$$

Need to find the limit of V_0^λ as $\lambda(\cdot) \to \infty$ (in some sense).

The Optimization Problem in Continuous Time

Denote by $\mathbb{F}^S$ the filtration generated by continuously observing S

$$\mathscr{F}_t^S := \sigma(S_u; 0 \leq u \leq t).$$

Consider the class $\mathscr{A}^c$ of continuous time strategies with short sale constraints: $\alpha = (\alpha_t)_{0 \leq t \leq T}$ such that

$$X_t = X_0 + \int_0^t \alpha_u \frac{dS_u}{S_{u-}}, \quad 0 \leq t \leq T$$

satisfies

$$0 \leq \alpha_t \leq X_{t-}, \quad 0 \leq t \leq T.$$

Define

$$V_0^M = \sup_{\alpha \in \mathscr{A}^c} \mathbb{E}[U(X_T)].$$

One Possible Approach

- solve the continuous time problem and find the optimal proportion $\hat{\pi}_s = \hat{\pi}(s, X_s) \in [0,1]$ in feedback form
- show that the solution of the closed loop equation

$$\hat{X}_t = X_0 + \int_0^t \hat{X}_u \hat{\pi}(u, \hat{X}_{u-}) \frac{dS_u}{S_{u-}}, \quad 0 \le t \le T$$

is approximated by the discrete version

$$X_{\tau_n} = X_{\tau_{n-1}} + \hat{X}_{\tau_{n-1}} \hat{\pi}(\tau_{n-1}, X_{\tau_{n-1}}) \frac{S_{\tau_n} - S_{\tau_{n-1}}}{S_{\tau_{n-1}}}, \quad n \ge 1.$$

when the intensity rate is large (at all times) so that

$$V_0^\lambda \to V_0^M$$

Our Approach

- for a fixed arrival rate λ, we use the very same verification arguments to show that an investor cannot improve his/her utility if the process S is observed continuously
- use the arguments of Kardaras and Platen to show that a continuous time trading strategy can be approximated by a discrete-time trading strategy (trading occurs at the arrival times) but where the asset S is observed continuously. Therefore
$$V_0^M \leq \lim_{\lambda} V_0^{\lambda}$$
- using independence $V_0^{\lambda} \leq V_0^M$, so
$$V_0^M = \lim_{\lambda} V_0^{\lambda}$$

Asymptotic behavior: the precise result

Consider $(\lambda_k)_k$ a sequence of intensity functions. If

$$\sum_{k=0}^{\infty} \exp\left(-\int_t^s \lambda_k(u)\,du\right) < \infty, \quad (\forall)\ \ 0 \leq t < s < T,$$

then

$$V_0^{\lambda_k} \to V_0^M, \quad \text{as } k \text{ goes to infinity.}$$

Conclusions

We model a time-illiquid investment problem as a stochastic control problem which is

- finite horizon in time t
- infinite horizon with respect to the number of trades/observations n

We solve the problem using dynamic programming

- solve the DPE
- perform a verification argument to find the optimal control

We also analyze the asymptotic behavior for the case when intensity is large at all times.