What is the problem
Risk-free rate puzzle
Equity premium puzzle

Continuous equilibria in incomplete markets

Kasper Larsen
Carnegie Mellon University (USA)

Joint work with

Peter Ove Christensen and Claus Munk
Aarhus University (DENMARK)

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The individual investor's problem

- Investor i, $i = 1,2,...,I$, is modeled by
  - Beliefs: $\mathbb{P}^{(i)}$ - probability measure on $\mathcal{F}_T$
  - Preferences: $U^{(i)}(c,t)$ or $U^{(i)}(x)$ - utility function
  - Income: $Y_t^{(i)}$ - income or endowment
- The financial market is modeled by
  - Money market account $S_t^{(0)}$: r_t is the interest rate
  - Dividends D_t: stochastic dividend rate process
  - Stock S_t: pays out the running dividend D_t
- Given these quantities, how should the investor optimally consume (c_t) and invest (θ_t) in order to maximize expected utility?
- Main obstacle: Market completeness - or rather the lack of it

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Utility maximization with income

- The problem without income, $Y := 0$, has been studied in a very general setting and we now know basically everything
- If the income process Y is spanned by the market $(S^{(0)}, S)$, the problem can be treated as a no income problem: simply increase the investor's initial endowment appropriately
- For unspanned income, the individual investor's problem is hard
    - Existence and uniqueness: Cvitanić, Schachermayer and Wang (2001), Karatzas and Žitković (2003), Žitković (2005)
- There are only few concrete examples
    - For exponential utility, see Henderson (2005)

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Equilibrium

- How do the equilibrium price processes $S_t^{(0)}$ and S_t look like?
    - The aggregate demand for $S^{(0)}$ - money market - has to be zero
    - The aggregate demand for the stock S has to equal the supply
    - All investors have to follow utility maximizing strategies
- Existence and uniqueness
    - The complete case is well-studied: the martingale method explicitly gives us the individual demand functions
    - The incomplete case is much less developed: several talks to come
- Examples of tractable models
    - In the complete case the representative agent can be used
    - Infinite horizon discrete time models with a continuum of agents with homogeneous preferences: see Constantinides and Duffie (1996) and Wang (2003)

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Clearing conditions

- We need to find $(S^{(0)}, S)$, i.e., we need the interest rate r, the drift μ_S and the volatility σ_S, such that

- Clearing in the stock market

$$\sum_{i=1}^{I} \hat{\theta}_t^{(i)} = \text{stock supply} \quad (\in \{0,1\})$$

- Clearing in the money market (zero supply)

$$\sum_{i=1}^{I} \hat{\rho}_t^{(i)} = 0$$

- Clearing in the goods market

$$\sum_{i=1}^{I} \hat{c}_t^{(i)} = D_t + \sum_{i=1}^{I} Y_{it}$$

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Exogenous parameters

- Identical beliefs (probabilities), $\mathbb{P}^{(i)} = \mathbb{P}$ for all i
- Linear dividend rate dynamics

$$dD_t := \mu_D(t)dt + \sigma_D(t)dW_t$$

- CARA preferences over running consumption

$$U^{(i)}(c,t) := -\exp(-\delta^{(i)}t - a^{(i)}c)$$

- We allow for different values of
  - Time preference parameter: $\delta^{(i)} \geq 0$
  - Risk preference parameter (absolute aversion): $a^{(i)} \geq 0$
- Linear income rate dynamics

$$dY_t^{(i)} := \mu_Y^{(i)}(t)dt + \sigma_Y^{(i)}(t)\left(\rho^{(i)}dW_t + \sqrt{1-\rho^{(i)}}dZ_t^{(i)}\right)$$

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Individual optimization problems

- The investor chooses a strategy θ to be invested in S (where the investment ρ in $S^{(0)}$ is given by the self-financing condition) and generates the wealth dynamics

$$dX_t^{(c,\theta)} := (X_t^{(c,\theta)} - \theta_t S_t) r_t dt + \theta_t (dS_t + D_t dt) + (Y_t - c_t) dt, \quad X_0^{(c,\theta)} := x$$

- Maximize expected utility over running consumption

$$u^{(i)}(x) := \sup_{(\theta,c) \in \mathcal{A}^{(i)}} \mathbb{E}\left[\int_0^T U^{(i)}(c_u, u) du \right],$$

where $(\theta, c) \in \mathcal{A}^{(i)}$ if $X_T^{(c,\theta)} \geq 0$ (plus integrability requirements)

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Equilibrium (à la Bachelier)

- The money market account $S^{(0)}$ has dynamics

$$dS_t^{(0)} = S_t^{(0)} r(t) dt,$$

  for a deterministic interest rate r

- The risky security S (in unit supply) has dynamics

$$dS_t = \Big(S_t r(t) + \mu_S(t) - D_t\Big) dt + \sigma_S(t) dW_t,$$

  for a deterministic drift μ_S and a deterministic vol σ_S

1. Conjecture the above form and solve each investor's problem, i.e., find the optimal number of shares $\hat{\theta}_t^{(i)}$ and the optimal consumption process $\hat{c}_t^{(i)}$ for $i = 1,...,I$
2. Aggregation produces dynamics consistent with conjecture

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Risk-free rate puzzle

- This puzzle is based on the observation that the historical risk-free rate is smaller than the risk-free rate predicted by consumption-based representative agent models with the high level of risk aversion needed to explain the high observed equity premium, see Weil (1989)

- Based on our equilibrium rate, we find the difference

$$r(t) - r^{\text{REP}}(t) = -\frac{1}{2}a^2(1-\rho^2)\sigma_Y^2,$$

where $I \to \infty$, $a^{(i)} = a$, $\rho^{(i)} = \rho$, $\sigma_Y^{(i)}(t) = \sigma_Y$

- This model does not produce an incompleteness effect on the equity premium

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Exogenous parameters

- CARA preferences of terminal wealth

$$U^{(i)}(x) := -\exp(-a^{(i)}x), \quad x \in \mathbb{R},$$

i.e., the investor seeks to maximize expected utility of terminal wealth $X_T^{(\theta)}$ plus random unspanned endowment $Y_T^{(i)}$:

$$u^{(i)}(x) := \sup_{\theta \in \mathcal{A}^{(i)}} \mathbb{E}[U^{(i)}(X_T^{(\theta)} + Y_T^{(i)})]$$

- Feller type vol process

$$dv_t := \left(\mu_v + \kappa_v v_t\right)dt + \sigma_v \sqrt{v_t}dW_t$$

- Investor i's random endowment process

$$dY_t^{(i)} := \mu_Y^{(i)}dt + \sqrt{v_t}\left(\sigma_Y^{(i)}dW_t + \beta_Y^{(i)}dZ_t^{(i)}\right),$$

where $Z^{(i)}$ is specific for investor i

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Equilibrium (à la Heston)

- Given investor *i* only receives utility of terminal wealth, we can take $r_t \triangleq 0$ in equilibrium.
- The risky security *S* has equilibrium dynamics

$$dS_t = \lambda_S(t)v_t dt + \sqrt{v_t}dW_t,$$

for a deterministic function λ_S
  - *S* is assumed to be in zero net supply and therefore there is no exogenously dividend for *S* paid at time *T* - the word 'stock' is therefore a bit inappropriate
  - Specifying a terminal exogenously divided payment is possible and nails down the vol structure of *S*
- The equilibrium market price of risk process

$$\lambda_S(t)\sqrt{v_t},$$

can be explicitly computed and depends on the incompleteness
- If $\sigma_v < 0$ (Gatheral) , this model produces $\lambda > \lambda^{\text{rep}}$