Financial equilibria in incomplete markets where heterogeneous agents with numéraire-invariant preferences act

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Outline

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A dynamic framework for preferences

Filtered probability space: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$.

- $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$: flow of information.
- $\mathbb{P}$ is a "baseline" (or "real world") probability.

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Quantities of interest: cumulative consumption streams. . .

- i.e., nondecreasing, (right-)continuous, adapted processes . . .
- whose densities with respect to some "consumption clock" H live on $(\Omega \times \mathbb{R}_+, \mathcal{O})$, where $\mathcal{O}$ is the *optional* sigma-algebra.

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Subjective views: unit-mass measures ("probabilities") on $(\Omega \times \mathbb{R}_+, \mathcal{O})$, generically denoted by $\mathcal{Q}$.

Numéraire-invariant choices on consumption streams

Utility from consumption has *logarithmic* form:

$$G \mapsto \int_{\Omega \times \mathbb{R}_+} \log \left(\frac{\mathrm{d}G}{\mathrm{d}H} \right) \mathcal{Q}[\mathrm{d}\omega, \mathrm{d}t]$$

for continuous nondecreasing *financeable* consumption streams G.

- Choice of H irrelevant for optimization: *numéraire-invariance*.

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First-order conditions imply that the optimal $\widehat{G}$ satisfies:

$$\mathsf{rel}_{\mathcal{Q}}(G \mid \widehat{G}) := \int_{\Omega \times \mathbb{R}_+} \left(\frac{\mathrm{d}G - \mathrm{d}\widehat{G}}{\mathrm{d}\widehat{G}} \right) \mathcal{Q}[\mathrm{d}\omega, \mathrm{d}t] \ \leq \ 0$$

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for all other *financeable* consumption streams G.

- Such choices stem from axiomatic foundations *à la Savage*. One then gets $\mathcal{Q}$ as a byproduct ...

Canonical representation of preferences

A decomposition for unit-mass optional measures:

$$\int_{\Omega \times \mathbb{R}_+} \left(\frac{\mathrm{d}G - \mathrm{d}\widehat{G}}{\mathrm{d}\widehat{G}} \right) \mathcal{Q}[\mathrm{d}\omega, \mathrm{d}t] = \mathbb{E}\left[\int_{\mathbb{R}_+} \left(\frac{\mathrm{d}G_t - \mathrm{d}\widehat{G}_t}{\mathrm{d}\widehat{G}_t} \right) L_t \mathrm{d}K_t \right]$$

where the canonical representation pair (L, K) of $\mathcal{Q}$ (i.e., of the numéraire-invariant preferences) is such that:

1. L is a nonnegative local martingale with $L_0 = 1$,
2. K is adapted, right-continuous, nondecreasing, $0 \leq K \leq 1$,

and it is essentially unique (under some "minimality" postulate).

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and it is essentially unique (under some "minimality" postulate).

Conversely: For (L, K) such that (1), (2), and $\mathbb{E}\left[\int_{\mathbb{R}_+} L_t\mathrm{d}K_t\right] = 1$ hold, one can construct a unit-mass optional measure $\mathcal{Q}$ such that (L, K) is its canonical representation pair.

Sometimes pairs are nice... but only sometimes.

Special case: If $L \equiv (\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})|_{\mathcal{F}_\cdot}$, then

$$\mathsf{rel}_{\mathcal{Q}}(G \mid \widehat{G}) \;=\; \mathbb{E}_{\mathbb{Q}}\left[\int_{\mathbb{R}_+} \left(\frac{\mathrm{d}G_t - \mathrm{d}\widehat{G}_t}{\mathrm{d}\widehat{G}_t}\right) \mathrm{d}K_t\right].$$

Remarks.

- Above, $\mathbb{Q}$ are the *subjective views* of the agent on $(\Omega, \mathcal{F})$.
- K: agent's *consumption clock*. We have $\mathbb{Q}[K_\infty = 1] = 1$.

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However,

- It *can* happen that L is *not* a martingale.
- It *can* also happen that $\mathbb{P}[K_\infty = 1] < 1$.

Outline

The market

Zero-net-supply asset: A *savings account* offering *instantaneous interest rate* $r = (r_t)_{t \in \mathbb{R}_+}$. The process B denotes the accrued value of one unit of account invested at time $t = 0$:

$$B = \exp\left(\int_0^{\cdot} r_t \mathrm{d}t\right) \Longrightarrow \frac{\mathrm{d}B_t}{B_t} = r_t \mathrm{d}t.$$

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Positive-net-supply asset: Offers per-share dividend rate process D and has price S with initial value S_0 and dynamics:

$$\frac{\mathrm{d}S_t + D_t \mathrm{d}t}{S_t} = (r_t + \alpha_t)\mathrm{d}t + \langle \sigma_t, \mathrm{d}W_t \rangle.$$

- $W = (W^n)_{n=1,\dots,m}$: standard BM that generates information.
- $|\sigma|$: local volatility process.
- α: excess rate of return.

Investment and consumption

Investment-consumption: with initial capital $x \in \mathbb{R}_+$, the control (π, c) generates wealth $X^{(x;\pi,c)}$ satisfying $X_0^{(x;\pi,c)} = x$ and

$$\frac{\mathrm{d}X_t^{(x;\pi,c)}}{X_t^{(x;\pi,c)}} = (1-\pi_t)\frac{\mathrm{d}B_t}{B_t} + \pi_t\left(\frac{\mathrm{d}S_t + D_t\mathrm{d}t}{S_t}\right) - c_t\mathrm{d}t$$

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Solution of the last linear SDE: $\log\left(X^{(x;\pi,c)}\right)$ is given by

$$\log(x) + \int_0^{\cdot}\left(r_t + \pi_t\alpha_t - \frac{\pi_t^2|\sigma_t|^2}{2} - c_t\right)\mathrm{d}t + \int_0^{\cdot}\pi_t\langle\sigma_t, \mathrm{d}W_t\rangle$$

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Consumption rate at $t \in \mathbb{R}_+$: $X_t^{(x;\pi,c)}c_t\mathrm{d}t$. Therefore, financeable consumption streams are of the form

$$G^{(x;\pi,c)} = \int_0^{\cdot} X_t^{(x;\pi,c)}c_t\mathrm{d}t.$$

Market viability

Minimal condition for optimization: $\{|\sigma| = 0\} \subseteq \{\alpha = 0\}$ and

$$\int_0^{\cdot} \left(\frac{\alpha_t}{|\sigma_t|} \mathbb{I}_{\{|\sigma_t| > 0\}} \right)^2 \mathrm{d}t$$

does not explode in finite time.

- Of interest is the case where $\{|\sigma| = 0\}$ has zero measure. (We search for a non-redundant equilibrium.)

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Market viability: The above condition fails *if and only if* there exists an arbitrage of the first kind in the market, i.e., some $\mathcal{F}_T$-measurable random variable ξ (for some $T \in \mathbb{R}_+$) such that:

- $\mathbb{P}[\xi \geq 0] = 1$, $\mathbb{P}[\xi > 0] > 0$, and
- for all $x > 0$, there exist π and c (that may depend on x) such that $\mathbb{P}[X_T^{(x;\pi,c)} = \xi] = 1$.

Outline

Numéraire-invariant choices

The market, agents, investment, consumption

Agent optimality

Equilibrium

Bubbles and arbitrages

Conclusions

Agent optimal investment-consumption

Preferences. Agent $i \in I$, where I is a finite set, has canonical representation pair (L^i, K^i), such that

$$\begin{aligned} \frac{\mathrm{d}K_t^i}{1 - K_t^i} &= \kappa_t^i \mathrm{d}t \\ \frac{\mathrm{d}L_t^i}{L_t^i} &= \langle \lambda_t^i, \mathrm{d}W_t \rangle \end{aligned}$$

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$$\frac{\mathrm{d}L_t^i}{L_t^i} = \langle \lambda_t^i, \mathrm{d}W_t \rangle$$

Optimality. The canonical representation pair (L^i, K^i) *separates* the solution to the problems of investment and consumption:

- the optimal portfolio is given by

$$\pi^i := \frac{\alpha + \langle \lambda^i, \sigma \rangle}{|\sigma|^2}.$$

- the optimal relative-to-wealth consumption rate satisfies:

$$c^i := \kappa^i.$$

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Market clearing and equilibrium

Several agents acting in the market, indexed by finite set I.

- $(L^i, K^i)_{i \in I}$ are the canonical preference pairs.
- $(x^i)_{i \in I}$: initial endowments, in terms of cash.
- $(\pi^i, c^i)_{i \in I}$: optimizers.
- Define $X^i := X^{(x^i;\pi^i,c^i)}$, for each $i \in I$.

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Equilibrium: in the market we have the clearing conditions

$$
\begin{aligned}
\sum_{i \in I} X^i &= S \quad \text{(money market)} \\
\sum_{i \in I} \frac{\pi^i X^i}{S} &= 1 \quad \text{(stock market)} \\
\sum_{i \in I} X^i c^i &= D \quad \text{(commodity market)}
\end{aligned}
$$

In search for equilibrium...

Primitives are exogenously given and include:

- canonical representation pairs $(L^i, K^i)_{i \in I}$ of agents.
- initial agent endowments; either cash and/or stock fractions.
- dividend structure D of asset in unit net supply.

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Outputs are sought in order to have market equilibrium.

- Short-rate structure: $(r_t)_{t \in \mathbb{R}_+}$.
- Stock price S; equivalently, S_0, $(\alpha_t)_{t \in \mathbb{R}_+}$ and $(\sigma_t)_{t \in \mathbb{R}_+}$.

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(Almost) necessary condition for equilibrium: Define

$$f^i := \frac{c^i}{D} = \frac{\kappa^i}{D}, \quad i \in I.$$

Assume that each f^i, $i \in I$, is an Itô process:

$$\frac{\mathrm{d}f_t^i}{f_t^i} = -\beta_t^i \,\mathrm{d}t - \langle \phi_t^i, \mathrm{d}W_t \rangle.$$

The state variables

State variables: $p = (p^i)_{i \in I}$, where, for $i \in I$,

$$p^i \; := \frac{X^i}{\sum_{j \in I} X^j} \; \left(= \frac{X^i}{S}, \text{ in equilibrium} \right).$$

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Notation. For $\zeta = (\zeta^i)_{i \in I}$, and $\mathrm{p} \in \triangle^I$, set $\zeta^{\mathrm{p}} := \sum_{i \in I} \mathrm{p}^i \zeta^i$.

Proposition. With the above notation, each p^i, $i \in I$, satisfies:

$$\begin{aligned} \frac{\mathrm{d}p^i_t}{p^i_t} &= \left(\frac{\langle \lambda^p_t - \lambda^i_t, \sigma_t \rangle \langle \lambda^p_t, \sigma_t \rangle}{|\sigma_t|^2} + \kappa^p_t - \kappa^i_t \right) \mathrm{d}t \\ &\quad + \frac{\langle \lambda^i_t - \lambda^p_t, \sigma_t \rangle}{|\sigma_t|^2} \langle \sigma_t, \mathrm{d}W_t \rangle. \end{aligned} \tag{SDE}$$

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Strategy. Our aim is the following:

- Express σ in terms of p and solve (SDE).
- Express all other market parameters in terms of p.

σ in terms of p: the idea

- Use the commodity clearing condition:

$$\sum_{i \in I} X^i c^i = D \iff \sum_{i \in I} p^i f^i = \frac{1}{S}.$$

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$$\left(1 + \frac{\langle \lambda^q - \lambda^p, \sigma \rangle}{|\sigma|^2}\right) \sigma = \phi^q,$$

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- where $q = (q^i)_{i \in I}$, each q^i for $i \in I$ being a function of p:

$$q^i = \frac{p^i f^i}{\sum_{j \in I} p^j f^j} = \frac{X^i c^i}{\sum_{j \in I} X^j c^j} \left(= \frac{X^i c^i}{D}, \text{ in equilibrium}\right).$$

σ in terms of p: implementation of idea

Solution to the equation for σ. Recall:

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$$\sigma = \left(1 + \frac{\langle \lambda^p - \lambda^q, \phi^q \rangle}{|\phi^q|^2}\right) \phi^q \qquad \text{(SIGMA)}$$

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- If $\phi^q = 0$,

$$\sigma \in \left\{ \langle \lambda^p - \lambda^q, \mathrm{u} \rangle \mathrm{u} \;\middle|\; |\mathrm{u}| = 1 \right\} \qquad \text{(SIGMA')}$$

α in terms of p

Use the stock clearing condition:

$$\sum_{i \in I} \pi^i X^i = S \iff \sum_{i \in I} \pi^i p^i = 1$$

to get:

$$\alpha = |\sigma|^2 - \langle \lambda^p, \sigma \rangle, \tag{ALPHA}$$

- σ has already been expressed in terms of p.

r in terms of p

Use again the commodity clearing condition

$$\sum_{i \in I} X^i c^i = D \iff \sum_{i \in I} p^i f^i = \frac{1}{S}.$$

- S in expressed in terms of p.

r in terms of p

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- S in expressed in terms of p.
- Now, equate the "$\mathrm{d}t$" parts above, remembering that

$$\frac{\mathrm{d}S_t + D_t \mathrm{d}t}{S_t} = (r_t + \alpha_t)\mathrm{d}t + \langle \sigma_t, \mathrm{d}W_t \rangle.$$

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- This way you *easily* get:

$$r = \ldots \tag{R}$$

Synthesis

The previous *necessary* conditions for equilibrium can be inverted:

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Theorem. If you do all the previous, and everything works out nicely, equilibrium is fully characterized.

- Solve (SDE) — *make sure it has a solution!*
- Define σ via (SIGMA) or (SIGMA');
- Define α via (ALPHA);
- Define r via (R).

Then, the market is in equilibrium.

Existence and uniqueness

The case $\phi^q \neq 0$: With $\widetilde{\phi}^q = (1/|\phi^q|)\phi^q$, for $i \in I$,

$$
\begin{aligned}
\frac{\mathrm{d}p_t^i}{p_t^i} &= \Big(\langle \lambda_t^p - \lambda_t^i, \widetilde{\phi}_t^q \rangle \langle \lambda_t^p, \widetilde{\phi}_t^q \rangle + \kappa_t^p - \kappa_t^i \Big)\,\mathrm{d}t \\
&+ \langle \lambda_t^i - \lambda_t^p, \widetilde{\phi}_t^q \rangle \langle \widetilde{\phi}_t^q, \mathrm{d}W_t \rangle.
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Existence and uniqueness of equilibrium becomes the equivalent question on the above equation. As long as ϕ^q does not vanish, it should be OK... but rigorous results have to be obtained.

Multiplicity of equilibria

The case $\phi = 0$: Recall (SIGMA'):

$$\sigma \in \left\{ \langle \lambda^p - \lambda^q, \mathrm{u} \rangle \mathrm{u} \;\middle|\; |\mathrm{u}| = 1 \right\}.$$

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Pick any predictable process $(u_t)_{t \in \mathbb{R}_+}$ with $|u| = 1$, and solve

$$\begin{aligned} \frac{\mathrm{d}p_t^i}{p_t^i} &= \left(\langle \lambda_t^p - \lambda_t^i, u_t \rangle \langle \lambda_t^p, u_t \rangle + \kappa_t^p - \kappa_t^i \right) \mathrm{d}t \\ &+ \langle \lambda_t^i - \lambda_t^p, u_t \rangle \langle u_t, \mathrm{d}W_t \rangle. \end{aligned}$$

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We can construct multiple *non-redundant* equilibria if:

- we have $m > 1$ (in particular, incompleteness);
- agents are "sufficiently heterogeneous";
- consumption rates and the dividend rate stream are smooth.

Special cases of closed-form equilibria

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Agents with same consumption clocks: In this case, $\sigma = \phi^p = \phi^i$ and (SDE) can be solved with a trick.

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"Folklore" knowledge: in non-constrained, complete-market equilibrium, assets in positive net supply cannot contain bubbles. (Otherwise, the *representative agent* would not invest in stocks.)

Stock bubble in equilibrium

Agent's preferences are given by the pair (L, K) such that:

- $\mathrm{d}L_t = -L_t^2 \mathrm{d}W_t$, $L_0 = 1$. (BES^{-3}, a *strict local martingale*.)
- $K = 1 - \exp\left(-\int_0^{\cdot} L_t^2 \mathrm{d}t\right)$.

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The stock contains a bubble, because $X^{(1;1,0)} = L$. For all $T \in \mathbb{R}_+$, there exists $x \in (0, 1)$ and strategy π, both depending on T, such that $\mathbb{P}\left[X_T^{(x;\pi,0)} = X_T^{(1;1,0)}\right] = 1$.

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- The state variable is the capital distribution. This seems to prevent Markovian structure in our equilibrium models...

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- Updating views and consumption patterns: $(L^i, K^i)_{i \in I}$ can be made to depend on past economy data.
  - past personal performance (learning from *your* mistakes),
  - past performance of other agents (learning from *their* mistakes or success; reevaluating statistical views),
  - consumption patterns of others (keeping up with the Johnses).

This way, *game-theoretic* equilibria can be constructed.

The End

THANK YOU!