

# **g-expectations and the representation of the penalty term of dynamic convex risk measures**

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Workshop on Foundations of Mathematical Finance – Toronto 2010

## Outline

- preliminaries and recalls on dynamic convex risk measures
- link with the existing literature
- representation of the penalty term of dynamic convex risk measures (or concave utilities) satisfying some suitable properties
- sketch of the proof (by applying the theory of BSDE and of  $g$ -expectations)

## Preliminaries

static coherent/convex risk measures are functionals (satisfying suitable axioms) quantifying now the riskiness of a financial position

in particular:

$$\rho : \mathcal{X} \rightarrow \mathbb{R}$$

where  $\mathcal{X}$  space of financial positions with maturity  $T$  (e.g.  $\mathcal{X} = L^\infty(\mathcal{F}_T)$ )

see Artzner et al. (1999), Delbaen (2002), Föllmer and Schied (2002), Frittelli and RG (2002)

Kusuoka, Frittelli and RG, Jouini, Schachermayer and Touzi, ...

dynamic risk measures quantify the riskiness of a position  $\xi$  of maturity  $T$  at any time  $s \in [0, T]$ :

$$\rho_s : \mathcal{X} \rightarrow L^0(\mathcal{F}_s)$$

in general:  $\rho_{s,t}(\xi)$  (with  $0 \leq s \leq t \leq T$ ) represents the riskiness of  $\xi$  (position of maturity  $t$ ) at time  $s$

see Artzner et al. (2007), Barrieu and El Karoui (2005), Cheridito, Delbaen and Kupper (2006), Delbaen (2006), Detlefsen and Scandolo (2005), Föllmer and Penner (2006), Frittelli and RG (2004), Klöppel and Schweizer (2007), RG (2006), ...

we will call *dynamic convex risk measure* on  $L^\infty$  :

$(\rho_{\sigma,\tau})_{0 \leq \sigma \leq \tau \leq T}$  such that, given any pair of stopping times  $\sigma$  and  $\tau$  such that  $0 \leq \sigma \leq \tau \leq T$ , the functional  $\rho_{\sigma,\tau} : L^\infty(\mathcal{F}_\tau) \rightarrow L^\infty(\mathcal{F}_\sigma)$  satisfies  $\rho_{\sigma,\tau}(0) = 0$  and

- **monotonicity**: if  $\xi, \eta \in L^\infty(\mathcal{F}_\tau)$  and  $\xi \leq \eta$ , then  $\rho_{\sigma,\tau}(\xi) \geq \rho_{\sigma,\tau}(\eta)$
- **translation invariance**:  $\rho_{\sigma,\tau}(\xi + \eta) = \rho_{\sigma,\tau}(\xi) - \eta$  for any  $\xi \in L^\infty(\mathcal{F}_\tau)$  and  $\eta \in L^\infty(\mathcal{F}_\sigma)$
- **convexity**:  $\rho_{\sigma,\tau}(\alpha\xi + (1 - \alpha)\eta) \leq \alpha\rho_{\sigma,\tau}(\xi) + (1 - \alpha)\rho_{\sigma,\tau}(\eta)$  for any  $\xi, \eta \in L^\infty(\mathcal{F}_\tau)$  and  $\alpha \in [0, 1]$

## Setting and hypothesis

$(B_t)_{t \geq 0}$   $d$ -dimensional Brownian motion

$(\mathcal{F}_t)_{t \geq 0}$  augmented filtration generated by  $(B_t)_{t \geq 0}$

$T > 0$  fixed finite time horizon

$L^\infty(\mathcal{F}_t)$  as space of risky positions with maturity  $t \in [0, T]$

we will identify any probability measure  $Q \sim P$  with its Radon  
- Nykodem density  $\frac{dQ}{dP}$

and with the ( $d$ -dimensional) predictable process  $(q_t)_{t \in [0, T]}$  induced by

$$E_P \left[ \frac{dQ}{dP} \middle| \mathcal{F}_t \right] = \mathcal{E}(q.B)_t \triangleq \exp \left( -\frac{1}{2} \int_0^t \|q_s\|^2 ds + \int_0^t q_s dB_s \right).$$

## Assumptions on the dynamic convex risk measures

**(A)**  $(\rho_{\sigma,\tau})_{0 \leq \sigma \leq \tau \leq T}$  is **continuous from above**, i.e. for any  $(\xi_n)_{n \in \mathbb{N}}$  in  $L^\infty(\mathcal{F}_\tau)$  such that  $\xi_n \downarrow \xi$  it holds  $\lim_n \rho_{\sigma,\tau}(\xi_n) = \rho_{\sigma,\tau}(\xi)$ .

**(B)**  $(\rho_{\sigma,\tau})_{\sigma,\tau}$  is **time-consistent**, i.e. for any stopping times  $\sigma, \tau, v$  with  $0 \leq \sigma \leq \tau \leq v \leq T$ :

$$\rho_{\sigma,v}(\xi) = \rho_{\sigma,\tau}(-\rho_{\tau,v}(\xi)), \quad \forall \xi \in L^\infty(\mathcal{F}_v).$$

**(C)**  $(\rho_{\sigma,\tau})_{\sigma,\tau}$  is **regular**, i.e.

$$\rho_{\sigma,\tau}(\xi 1_A + \eta 1_{A^c}) = \rho_{\sigma,\tau}(\xi) 1_A + \rho_{\sigma,\tau}(\eta) 1_{A^c}, \quad \forall \xi, \eta \in L^\infty(\mathcal{F}_\tau), \forall A \in \mathcal{F}_\sigma.$$

**(D)**  $c_{t,T}(P) = \text{ess sup}_{\xi \in L^\infty(\mathcal{F}_T)} \{E_P[-\xi | \mathcal{F}_t] - \rho_{t,T}(\xi)\} = 0, \forall t$

## Remarks on the Assumptions

- (A) is the dynamic version of the static lsc
- (B) is the key assumption. By Kupper and Schachermayer (2008), (B) + law invariance imply that  $\exists \gamma \in [0, +\infty]$  s.t.

$$\rho_{t,T}(\xi) = \frac{1}{\gamma} \ln E [\exp\{-\gamma X\} | \mathcal{F}_t]$$

i.e. either entropic-type or worst case risk measure

- it is sufficient to suppose the existence of  $Q \sim P$  satisfying (D).

$c_{t,T}$  as in Assumption (D) is the minimal penalty term associated to  $\rho_{t,T}$

## Some known results in the literature

- (see Bion-Nadal (2006); Detlefsen and Scandolo (2005))

If  $(\rho_{\sigma,\tau})_{0 \leq \sigma \leq \tau \leq T}$  satisfies the assumptions above, then

$$\rho_{s,t}(\xi) = \text{ess.} \sup_{Q \ll P, Q=P \text{ on } \mathcal{F}_s} \{E_Q[-\xi | \mathcal{F}_s] - c_{s,t}(Q)\} \quad (1)$$

for any  $0 \leq s \leq t \leq T$ , where

$$c_{s,t}(Q) = \text{ess.} \sup_{\xi \in L^\infty(\mathcal{F}_t)} \{E_Q[-\xi | \mathcal{F}_s] - \rho_{s,t}(\xi)\}$$

- since  $c_{t,T}(P) = 0$ : in (1)  $Q \ll P$  may be replaced by  $Q \sim P$   
(see Klöppel and Schweizer (2007) (continuous time), Föllmer and Penner, Cheridito et al. (discrete time))
- (see Bion-Nadal (2006), Föllmer and Penner (2006))  
time-consistency is equivalent to the *cocycle property* of the penalty term  $c$ , i.e.

$$c_{\sigma,v}(Q) = c_{\sigma,\tau}(Q) + E_Q [c_{\tau,v}(Q) | \mathcal{F}_\sigma]$$

for any stopping times  $\sigma, \tau, v$  such that  $0 \leq \sigma \leq \tau \leq v \leq T$

## NOTATIONS:

$$c_t(Q) = c_{t,T}(Q)$$

$$\rho_t(X) = \rho_{t,T}(X)$$

Our main result is the following.

**Theorem 1** *Let  $(\rho_{\sigma,\tau})_{0 \leq \sigma \leq \tau \leq T}$  be a dynamic convex risk measure*

*Then: for any stopping times  $\sigma, \tau$  such that  $0 \leq \sigma \leq \tau \leq T$*

$$c_{\sigma,\tau}(Q) = E_Q \left[ \int_{\sigma}^{\tau} f(u, q_u) du \middle| \mathcal{F}_{\sigma} \right] \quad (2)$$

*for some function  $f : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow [0, +\infty]$  such that  $f(t, \omega, \cdot)$  is proper, convex and lower semi-continuous.*

The statement of Theorem 1 can be “translated” for **dynamic concave utilities**  $(u_{\sigma,\tau})_{0 \leq \sigma \leq \tau \leq T}$ , where  $u_{s,t}(\xi) \triangleq -\rho_{s,t}(\xi)$ .

The proof of the previous representation is based on the theory of BSDE and on  $g$ -expectations.

**Basic references** (Lipschitz condition on  $g$ ):

Pardoux and Peng (1990), Peng (1997), El Karoui, Peng and Quenez (1997), Coquet et al. (2001, 2002), Peng (2004), ...

**Recent references** (weaker conditions on  $g$ ):

Lepeltier and San Martin (1998), Kobylansky (2000), Briand and Hu (2006, 2008), Delbaen, Hu and Bao (2009), ...

Let

$$\begin{aligned} g : \mathbb{R}^+ \times \Omega \times \mathbb{R} \times \mathbb{R}^d &\rightarrow \mathbb{R} \\ (t, \omega, y, z) &\longmapsto g(t, \omega, y, z) \end{aligned}$$

be a functional such that

- $g$  is (uniformly) Lipschitz in  $(y, z)$
- $g(\cdot, y, z)$  is predictable and such that  $E \left[ \int_0^s (g(t, \omega, y, z))^2 dt \right] < +\infty$  for any  $s > 0$
- $(dt \times dP)$ -a.s.,  $\forall y \in \mathbb{R}, g(t, y, 0) = 0$ .

Given  $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ , the following Backward Stochastic Differential Equation:

$$\begin{cases} -dY_t = g(t, Y_t, Z_t)dt - Z_t dB_t \\ Y_T = \xi, \end{cases}$$

has a unique solution  $(Y_t, Z_t)_{t \in [0, T]}$  of predictable stochastic processes such that  $E[\int_0^T Y_t^2 dt] < +\infty$  and  $E[\int_0^T \|Z_t\|^2 dt] < +\infty$  (see Pardoux and Peng (1990)).

Given such a solution  $(Y_t, Z_t)_{t \in [0, T]}$ , Peng (1997) defined

$$\mathcal{E}_g(\xi) \triangleq Y_0 \quad \text{g-expectation of } \xi$$

$$\mathcal{E}_g(\xi | \mathcal{F}_t) \triangleq Y_t \quad \text{conditional g-expectation of } \xi$$

Particular cases:

- when  $g = 0$ :  $\mathcal{E}_g(\cdot | \mathcal{F}_t) = E_P(\cdot | \mathcal{F}_t)$
- when  $g(t, y, z) = \mu \|z\|$ ,  $\mu > 0$ :  $\mathcal{E}_g(\cdot | \mathcal{F}_t)$  denoted by  $\mathcal{E}^\mu(\cdot | \mathcal{F}_t)$

the properties satisfied by  $\mathcal{E}_g(\cdot|\mathcal{F}_t)$  depend on the assumptions imposed on  $g$ :

- $g$  does not depend on  $y \Leftrightarrow \mathcal{E}_g(\cdot|\mathcal{F}_t)$  translation invariant
- $g$  convex in  $(y, z) \Leftrightarrow \mathcal{E}_g(\cdot|\mathcal{F}_t)$  convex
- $g$  positive homogeneous in  $(y, z) \Leftrightarrow \mathcal{E}_g(\cdot|\mathcal{F}_t)$  PH

(see Peng (1997), El karoui, Peng and Quenez (1997), RG (2006) and Jiang (2008))

What is the link between risk measures and  $g$ -expectations?

(i)  $\rho_{t,T}^g(X) = \mathcal{E}_g(-X|\mathcal{F}_t)$  is a time-consistent risk measure

(ii) under suitable assumptions the converse is also true

(see RG (2006), Barrieu and El Karoui, ...)

Where  $g$ -expectations play a role in our result?

... in a while ...

... coming back to our result and its proof ...

## Sketch of the proof of Theorem 1

Step 1:

Set

$$\rho_{s,t}^n(\xi) \triangleq \text{ess.} \sup_{\substack{Q \sim P; \|q\| \leq n \\ Q = P \text{ on } \mathcal{F}_s}} \{E_Q[-\xi|\mathcal{F}_s] - c_{s,t}(Q)\}.$$

Then  $\rho^n$  is a dynamic convex risk measure satisfying the assumptions above. Moreover:

$$c_{s,t}^n(Q) = \begin{cases} c_{s,t}(Q); & \text{if } \|q\| \leq n \\ +\infty; & \text{otherwise} \end{cases}$$

**Remark** By Assumption (D) ( $c_t(P) = 0$ ):

$$\rho_t^n(\xi) \geq E_P[-\xi|\mathcal{F}_t]$$

$$\rho_t^0(\xi) = E_P[-\xi|\mathcal{F}_t]$$

**BASIC IDEA:**

to prove

$$c_{s,t}^n(Q) = E_Q \left[ \int_s^t f_n(u, q_u) du | \mathcal{F}_s \right]$$

for suitable  $f_n$  and to pass somehow to the limit ...

Step 2:  $\rho^n$  is induced by a conditional  $g_n$ -expectation, i.e.

$$\rho_t^n(\xi) = \mathcal{E}_{g_n}(-\xi|\mathcal{F}_t) \quad (3)$$

with  $g_n$  convex, lsc functional, Lipschitz of constant  $n$  (in  $z$ ) and satisfying the usual hypothesis.

Hence:  $\rho^n$  satisfies the following BSDE

$$\begin{cases} -d\rho_t^n(\xi) &= g_n(t, Z_t^n)dt - Z_t^n dB_t \\ \rho_T^n(\xi) &= -\xi \end{cases}$$

## Step 2 (cont.):

- Conditional  $g_n$ -expectation: by a result of Coquet et al. (2002), it is sufficient to verify that

$$\pi_t^n(\xi) \triangleq \rho_{t,T}^n(-\xi)$$

is time-consistent and that  $\pi_0^n$  satisfies strict monotonicity, translation invariance, constancy and  $\mathcal{E}^\mu$ -dominance.

$$\left[ \begin{array}{l} \pi \text{ is } \mathcal{E}^\mu\text{-dominated if:} \\ \exists \mu > 0 \text{ s.t. } \pi(X + Y) - \pi(X) \leq \mathcal{E}^\mu(Y) \text{ for any } X, Y \end{array} \right]$$

- Convexity of  $g_n$ : consequence of a result of Jiang (2008)

Step 3:

$$c_{s,t}^n(Q) = E_Q \left[ \int_s^t f_n(u, q_u) du \middle| \mathcal{F}_s \right] \quad (4)$$

where  $f_n(t, \omega, \cdot) = (g_n(t, \omega, \cdot))^*$ .

For  $s = 0, t = T$ , equation (4) can be deduced by  $c_0^n(Q) = (\pi_0^n)^*(Q)$  and by the Measurable Selection Theorem.

A dual representation of  $g$ -expectations also in Barrieu and El Karoui (2005).

The general case can be obtained thanks to the cocycle property of  $c$ .

## Some remarks

- $f_n(t, \omega, q) \geq 0$  (since  $g_n(t, \omega, 0) = 0$ )
- $f_n(t, \omega, 0) = 0$  (from  $c_0(P) = 0$  and  $g_n(t, \omega, z) \geq 0$ )
- $f_n(t, \omega, q) = +\infty$  for  $\|q\| > n$

#### Step 4:

- $c_0^n$  is decreasing on  $n$
- $\rho^n$  is increasing on  $n$ , hence  $\mathcal{E}_{g_n}(\xi|\mathcal{F}_t) \leq \mathcal{E}_{g_{n+1}}(\xi|\mathcal{F}_t)$
- By applying the Converse Comparison Theorem on BSDE (see Briand et al. (2000)) and a result of Jiang (2006):

$g_n$  is increasing on  $n$

- $\rightsquigarrow f_n$  is decreasing on  $n$

Step 4 (cont.):

the sequence of  $f_n$  “stabilizes”, that is, once fixed  $(t, \omega)$ ,  $\forall q$

either: there exists  $n \geq 0$  such that  $f_m(t, \omega, q) = f(t, \omega, q) < +\infty$   
for any  $m \geq n$

or:  $f_n(t, \omega, q) = +\infty = f(t, \omega, q)$  for any  $n \geq 0$

for some function  $f : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow [0, +\infty]$ .

Therefore:  $f(t, \omega, x) = \inf_n f_n(t, \omega, x)$  is proper, convex and lower semi-continuous and  $f(t, \omega, 0) = 0$ .

Step 5:

Case 1:  $\int_0^T f(t, q_t) dt$  bounded

Set  $Q^n$  the probability associated to  $q^n = q \mathbf{1}_{\|q\| \leq n}$ .

By lsc of  $c$ :

$$\begin{aligned} c_{0,T}(Q) &\leq \lim_n c_{0,T}(Q^n) \\ &= \lim_n E_{Q^n} \left[ \int_0^T f_n(t, q_t^n) dt \right] \\ &= E_{Q^n} \left[ \int_0^T f(t, q_t) \mathbf{1}_{\|q\| \leq n} dt \right] \\ &= E_Q \left[ \int_0^T f(t, q_t) dt \right] \end{aligned}$$

Step 5 (cont.):

Case 2:  $\int_0^T f(t, q_t) dt$  unbounded

... similarly ...

Hence: for any  $Q \sim P$

$$c_{0,T}(Q) \leq E_Q \left[ \int_0^T f(u, q_u) du \right]$$

$$c_{t,T}(Q) \leq E_Q \left[ \int_t^T f(u, q_u) du \middle| \mathcal{F}_t \right]$$

??? is it possible to replace  $\leq$  with  $=$  ???

IDEA / AIM:

to find (if there exists) a sequence  $(Q^m)_{m \geq 0}$  with bounded  $q^m$  such that  $\frac{dQ^m}{dP} \rightarrow^{L^1} \frac{dQ}{dP}$  and  $c_{0,T}(Q^m) \rightarrow_m c_{0,T}(Q)$ . Hence, by proceeding as above:

$$\begin{aligned} c_{0,T}(Q) &\leq E_Q \left[ \int_0^T f(u, q_u) du \right] \\ &\leq \dots \\ &\leq \lim_m E_{Q^m} \left[ \int_0^T f(u, q_u) \mathbf{1}_{\|q\| \leq m} du \right] \\ &= \lim_m c_{0,T}^m(Q^m) = c_{0,T}(Q). \end{aligned}$$

## Step 6:

(a)  $c_t$  is a positive  $Q$ -supermartingale of class (D)

(b) if  $Q \sim P$ ,  $c_{0,T}(Q) < +\infty$  and  $c_t(Q)$  is right-continuous, then  $c_t(Q)$  is a  $Q$ -potential,

i.e. there exists a unique predictable, increasing  $(A_t^Q)_{t \in [0,T]}$  such that  $A_0^Q = 0$  and

$$c_t(Q) = E_Q[A_T^Q - A_t^Q | \mathcal{F}_t]$$

(by Dellacherie-Meyer)

(c)  $(c_t(Q))_{t \in [0,T]}$  is càdlàg  $\Rightarrow (A_t^Q)_{t \in [0,T]}$  is càdlàg

### Step 7:

(a) Given  $Q^1, Q^2$ , their associated processes  $A^1, A^2$  and two stopping times  $\sigma, \tau$ , set  $q \triangleq q^1 1_{H_1} + q^2 1_{H_2}$  where  $H_1 = ]0, \sigma] \cup ]\tau, T]$  and  $H_2 = ]\sigma, \tau]$ . Then

$$dA^Q = 1_{H_1} dA^1 + 1_{H_2} dA^2$$

(b) let  $Q \sim P$  and let  $A = A^Q$  be the increasing process associated to  $Q$ . Suppose  $A$  bounded.

Let  $H$  be a predictable set and let  $Q^H$  be the probability measure corresponding to  $q^H = q 1_H$ .

Then  $dA^H \leq dA$ , hence  $A_T^H \leq A_T$ .

Step 8:

(a) let  $Q \sim P$  and let  $A = A^Q$  be bounded.

Let  $(H^m)_{m \geq 0}$  be a sequence of predictable sets with  $H^m \uparrow (0, T] \times \Omega$  and let  $Q^{H^m}$  be the probability measure corresponding to  $q^{H^m} = q1_{H^m}$ .

Then  $c_{0,T}(Q^{H^m}) \rightarrow_m c_{0,T}(Q)$ .

(b) by taking  $H^m = \{q : \|q\| \leq m\}$ :

there exists  $(Q^m)_{m \geq 0}$  with bounded  $q^m$  such that  $\frac{dQ^m}{dP} \rightarrow^{L^1} \frac{dQ}{dP}$  and

$$c_{0,T}(Q^m) \rightarrow_m c_{0,T}(Q)$$

**Final step:** by the existence of a sequence  $(Q^m)_{m \geq 0}$  (Step 8), we deduce that

$$\begin{aligned} c_{0,T}(Q) &\leq E_Q \left[ \int_0^T f(u, q_u) du \right] \\ &\leq \dots \\ &\leq \liminf_m c_{0,T}^m(Q^m) \\ &= \liminf_m c_{0,T}(Q^m) = c_{0,T}(Q) \end{aligned}$$

Hence:

$$c_{0,T}(Q) = E_Q \left[ \int_0^T f(u, q_u) du \right]$$

... and by the cocycle property:

$$c_{\sigma,\tau}(Q) = E_Q \left[ \int_{\sigma}^{\tau} f(u, q_u) du \middle| \mathcal{F}_{\sigma} \right]$$

**End of the proof.**

A relevant example: the entropic penalty term

(see Barrieu and El Karoui (2005))

for  $f(q) = \frac{1}{2}\|q\|^2$ :

$$c_{0,T}(Q) = E_Q \left[ \int_0^T \frac{1}{2} \|q_t\|^2 dt \right] = H(Q, P)$$

... even if the corresponding  $g$  is not Lipschitz !

## Main references

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*THANK YOU*

*FOR YOUR ATTENTION !!!*