Pricing and Hedging of CDOs: A Top Down Approach

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Outline

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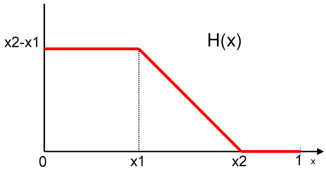
Single-Tranche Synthetic CDO (STCDO)

- Standard instrument for investing in CDO-pool (e.g. iTraxx)
- Specified by
  - coupon dates $0 < T_1 < \cdots < T_n$
  - a tranche with attachment/detachment points $x_1 < x_2$
  - a fixed swap rate s_0

Cash-flow of a STCDO

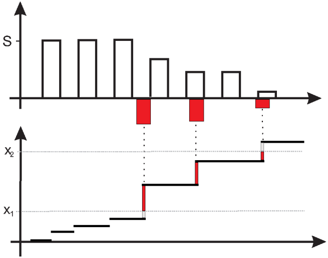
Cash-flow attributed to tranche $(x_1, x_2]$

- Denote aggregate loss process by L
- Define $H(x) = (x_2 - x)^+ - (x_1 - x)^+ = \int_{x_1}^{x_2} 1_{\{x \leq y\}} dy$



An investor in this STCDO

- is default protection seller,
- receives $s_0 \times H(L_{T_i})$ at T_i (premium leg),
- pays $-\Delta H(L_t) = H(L_{t-}) - H(L_t)$ at any time $t \in (T_0, T_n]$ where $\Delta L_t \neq 0$ (protection leg)



Example: iTraxx Europe

- Comprises CDS of 125 equally-weighted European entities
- Six tranches: 0–3, 3–6, 6–9, 9–12, 12–22, 22–100%
- Five maturities: 1, 3, 5, 7, 10 years
- Markit iTraxx CDS Indices, www.markit.com

Pricing and Hedging of STCDOs

- Index ($x_1 = 0$, $x_2 = 1$) corresponds to portfolio of constituent CDS

$\Rightarrow$ Can be replicated by holding constituent CDS

- Problem 1: price STCDOs
- Problem 2: hedge STCDO using the index

- Solution: develop top-down framework: (T, x)-bonds …

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Related Literature

- Bielecki–Crépey–Jeanblanc [BCJ08]
- Bielecki–Jeanblanc–Rutkowski [BJR07]
- Cont–Kan [CK08]
- Frey–Backhaus [FB07]
- ...

Outline

(T,x)-Bonds

- CDO pool of credits normalized to 1
- Loss process

$$L_t = \sum_{s\leq t} \Delta L_s = \int_0^t \int_{(0,1]} x\,\mu(ds,dx)$$

non-decreasing point process with compensator $\nu(t,dx)dt$

- (T,x)-bond pays

$$1_{\{L_T\leq x\}}$$

at maturity T, $x\in[0,1]$

- Price at t is proportional to $\mathcal{F}_t$-conditional $\mathbb{Q}^T$-cdf of L_T

$$P(t,T,x) = P(t,T)\mathbb{Q}^T\left[L_T \leq x \mid \mathcal{F}_t\right]$$

where $P(t,T) = P(t,T,1)$ risk-free zero-coupon bond

(T, x)-Bonds Spanning Property

- Contingent claim with payoff $F(L_T)$ at T can be decomposed

$$F(L_T) = F(1) - \int_0^1 F'(x) 1_{\{L_T \leq x\}} \, dx$$

- Hence static replicating portfolio, at $t \leq T$, is

$$F(1)P(t,T) - \int_0^1 F'(x) P(t,T,x) \, dx$$

$\Rightarrow$ (T, x)-bonds span all European type contingent claims

- Example: STCDOs based on $H(y) = \int_{x_1}^{x_2} 1_{\{y \leq x\}} dx$ ...

Term-Structure Movements

- Factorize (T, x)-bond price movements by

$$P(t, T, x) = \underbrace{1_{\{L_t \le x\}}}_{\text{default risk}} \underbrace{\mathrm{e}^{-\int_t^T f(t,u,x)\,du}}_{\text{market risk}}$$

- where (T, x)-forward rates $f(t, T, x)$ follows semimartingale

$$f(t, T, x) = f(0, T, x) + \int_0^t a(s, T, x)ds + \int_0^t b(s, T, x)^\top dW_s + \int_0^t \int_{(0,1]} c(s, T, x; y)\,\mu(ds, dy)$$

- Filipović–Overbeck–Schmidt [FOS09], Sidenius–Piterbarg–Andersen [SPA08]

(T, x)-Forward Rates

- risk-free forward rate $f(t, T) = f(t, T, 1)$
- risk-free short rate $r_t = f(t, t)$
- Can show:

$$f(t, T, x) - f(t, T) = \lim_{\Delta T \to 0} \frac{1}{\Delta T} \mathbb{Q}^T[L_{T+\Delta T} > x \mid L_T \leq x, \mathcal{F}_t]$$

$= \mathbb{Q}^T$-forward transition rate prevailing at date t for L to jump at T from below or equal to above level x

- cf. Ehlers–Schönbucher [ES06, ES09], Lipton–Shelton [LS09]: finite grid forward transition probabilities

No-Arbitrage Conditions [FOS09]

Theorem (No-Arbitrage)

$Z(t,T,x) = \mathrm{e}^{-\int_0^t r_s\,ds} P(t,T,x)$ local martingale for all (T,x) holds if and only if

$$\int_t^T a(t,u,x)\,du = \frac{1}{2}\left\|\int_t^T b(t,u,x)\,du\right\|^2 + \int_{(0,1]}\left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\,du} - 1\right) 1_{\{L_t+y\le x\}}\,\nu(t,dy),$$

$$\nu(t,(0,x]) = f(t,t,L_t) - f(t,t,L_t+x)$$

on $\{L_t \le x\}$, $dt \otimes d\mathbb{Q}$-a.s. for all (T,x).

Theorem (Existence)

For any nice $b(t,T,x)$ and $c(t,T,x)$ there exists an arbitrage-free pair $\{f(t,T,x), L_t\}$.

Risk-Neutral (T, x)-Bond Dynamics

Corollary

The risk-neutral dynamics of $Z(t, T, x) = \mathrm{e}^{-\int_0^t r_s\, ds} P(t, T, x)$ *is*

$$\frac{dZ(t, T, x)}{Z(t-, T, x)} = \beta(t, T, x)\, dW_t + \int_{(0,1]} \gamma(t, T, x, \xi)(\mu(dt, d\xi) - \nu(t, d\xi)dt)$$

where

$$\beta(t, T, x) = -\int_t^T b(t, u, x)^\top du$$

$$\gamma(t, T, x, \xi) = \mathrm{e}^{-\int_t^T c(t,u,x;\xi)\, du} 1_{\{L_{t-}+\xi \le x\}} - 1.$$

In particular $\frac{\Delta Z(t,T,x)}{Z(t-,T,x)} = \gamma(t, T, x, \Delta L_t) 1_{\{\Delta L_t > 0\}}$.

Outline

Recap

- Specified by:
  - Coupon dates $0 < T_1 < \cdots < T_n$
  - Attachment/detachment points $x_1 < x_2$
  - Fixed swap rate s_0
- An investor in this STCDO
  - Receives $s_0 \times H(L_{T_i})$ at T_i
  - Pays $-\Delta H(L_t) = H(L_{t-}) - H(L_t)$ at any $t \in (T_0, T_n]$

Accumulated Discounted Cash Flow

Integration by parts:

$$A_t = \underbrace{s_0 \sum_{T_i \le t} \mathrm{e}^{-\int_0^{T_i} r_s\, ds} H(L_{T_i})}_{\text{premium leg}} + \underbrace{\int_0^t \mathrm{e}^{-\int_0^u r_s\, ds}\, dH(L_u)}_{\text{protection leg}}$$

$$= \int_{(x_1,x_2]} \Bigg\{ s_0 \sum_{T_i \le t} \mathrm{e}^{-\int_0^{T_i} r_s\, ds} 1_{\{L_{T_i} \le x\}}$$

$$+ \mathrm{e}^{-\int_0^t r_s\, ds} 1_{\{L_t \le x\}} - 1_{\{L_0 \le x\}} + \int_0^t r_u \mathrm{e}^{-\int_0^u r_s\, ds} 1_{\{L_u \le x\}}\, du \Bigg\} dx$$

Assumption

$A_t \in L^2$ for all t.

Discounted Spot Value

$$
\begin{aligned}
\Gamma_t &= \mathbb{E}\left[A_{T_n} - A_t \mid \mathcal{F}_t\right] \\
&= \int_{(x_1,x_2]} \Big\{ s_0 \sum_{t<T_i} Z(t, T_i, x) \\
&\qquad - \mathrm{e}^{-\int_0^t r_s\, ds} 1_{\{L_t \le x\}} + Z(t, T_n, x) + \delta(t,x) \Big\} dx
\end{aligned}
$$

where

$$
\delta(t,x) = \int_t^{T_n} \mathbb{E}\left[r_u \mathrm{e}^{-\int_0^u r_s\, ds} 1_{\{L_u \le x\}} \mid \mathcal{F}_t \right] du
$$

Par Swap Rate

- Defined by $\Gamma_t = 0$:

$$s_t = \frac{\int_{(x_1,x_2]} \left\{ \mathrm{e}^{-\int_0^t r_s\, ds} 1_{\{L_t \le x\}} - Z(t, T_n, x) - \delta(t,x) \right\} dx}{\int_{(x_1,x_2]} \sum_{t<T_i} Z(t, T_i, x)\, dx}$$

- Consequence:

$$\Gamma_t = (s_0 - s_t) \int_{(x_1,x_2]} \sum_{t<T_i} Z(t, T_i, x)\, dx$$

Gains Process

$G_t = A_t + \Gamma_t$ is a square integrable martingale and satisfies

$$dG_t = \int_{(x_1,x_2]} \left\{ s_0 \sum_{t<T_i} dZ(t, T_i, x) + dZ(t, T_n, x) + r_t \mathrm{e}^{-\int_0^t r_s\, ds} 1_{\{L_t \le x\}}\, dt + d\delta(t, x) \right\} dx.$$

Notation

Write $s_t = s_t^{(x_1,x_2]}$, $G_t = G_t^{(x_1,x_2]}$, *etc.*

Outline

Variance-Minimizing Strategy

Theorem

The self-financing strategy

$$\phi^* = \frac{d\langle G^{(x_1,x_2]}, G^{(0,1]}\rangle}{d\langle G^{(0,1]}\rangle}$$

along with initial capital $c^* = G_t^{(x_1,x_2]}$ *is the unique minimizer of the quadratic hedging error*

$$\operatorname{essinf}_{c,\phi}\mathbb{E}\left[\left(c + \int_t^T \phi_s \, dG_s^{(0,1]} - G_T^{(x_1,x_2]}\right)^2 \mid \mathcal{F}_t\right].$$

Proof.

Goultchuk–Kunita–Watanabe decomposition of $G^{(x_1,x_2]}$. □

Møller [Møl01], Cont–Kan [CK08]

The δ-Term

- Problem with $\delta(t,x) = \int_t^{T_n} \mathbb{E}\left[r_u \mathrm{e}^{-\int_0^u r_s\,ds} 1_{\{L_u \le x\}} \mid \mathcal{F}_t\right] du$

Assumption

Deterministic short rates r_t

- Then $\delta(t,x) = \int_t^{T_n} r_u Z(t,u,x)\,du$

Gains Process Simplifies

$$dG_t^{(x_1,x_2]} = e^{-\int_0^t r_u du}\left(B_t^{(x_1,x_2]}\, dW_t + \int_{(0,1]} C_t^{(x_1,x_2]}\, d(\mu - \nu)\right)$$

where

$$B_t^{(x_1,x_2]} = \int_{(x_1,x_2]} \left\{ s_0^{(x_1,x_2]} \sum_{t<T_i} P(t, T_i, x)\beta(t, T_i, x) \right.$$
$$\left. + P(t, T_n, x)\beta(t, T_n, x) + \int_t^{T_n} r_u P(t, u, x)\beta(t, u, x)\, du \right\} dx$$
$$C_t^{(x_1,x_2]}(\xi) = \int_{(x_1,x_2]} \left\{ s_0^{(x_1,x_2]} \sum_{t<T_i} P(t-, T_i, x)\gamma(t, T_i, x, \xi) \right.$$
$$\left. + P(t-, T_n, x)\gamma(t, T_n, x, \xi) + \int_t^{T_n} r_u P(t-, u, x)\gamma(t, u, x, \xi)\, du \right\} dx.$$

Explicit Variance-Minimizing Strategy

$$\phi_t^* = \frac{B_t^{(x_1,x_2]} B_t^{(0,1]} - \int_{(0,1]} C_t^{(x_1,x_2]}(\xi) C_t^{(0,1]}(\xi) \, f(t,t,L_t + d\xi)}{(B_t^{(0,1]})^2 - \int_{(0,1]} (C_t^{(0,1]}(\xi))^2 \, f(t,t,L_t + d\xi)}$$

can be computed at t by

- the observables

$$s_0^{(x_1,x_2]}, \quad P(t,T,x), \quad r_T, \quad L_t$$

- and the model parameters

$$\beta(t,T,x), \quad \gamma(t,T,x,\cdot)$$

Outline

Affine Term Structure [FOS09]

- Affine term structure

$$\int_t^T f(t,s,x)\,ds = A(T-t,x) + B(T-t,x)Y_t$$

- Feller square root factor process

$$dY_t = (\mu_0 + \mu_1 Y_t)dt + \sigma\sqrt{Y_t}dW_t, \quad Y_0 = y \in \mathbb{R}_+$$

- Riccati equations

$$\partial_t A(t,x) = \alpha_0(x) + \mu_0 B(t,x)$$

$$\partial_t B(t,x) = \beta_0(x) + \mu_1 B(t,x) - \frac{\sigma^2}{2}B(t,x)^2$$

with $A(0,x)=0,\ B(0,x)=0$

Explicit Solution

$$A(t,x) = \alpha_0(x)t - \frac{2\mu_0}{\sigma^2}\log\left(\frac{2\rho(x)\mathrm{e}^{\frac{(\rho(x)-\mu_1)t}{2}}}{\rho(x)\left(e^{\rho(x)t}+1\right)-\mu_1\left(e^{\rho(x)t}-1\right)}\right)$$

$$B(t,x) = \frac{2\beta_0(x)\left(e^{\rho(x)t}-1\right)}{\rho(x)\left(e^{\rho(x)t}+1\right)-\mu_1\left(e^{\rho(x)t}-1\right)}$$

where $\rho(x) = \sqrt{\mu_1^2 + 2\sigma^2\beta_0(x)}$

Loss Intensity and LGD Distribution

- Exogenous parameters $\alpha_0(x)$, $\beta_0(x)$ non-increasing càdlàg
- $\alpha_0(x) \equiv r \geq 0$ constant risk-free rate, for $x \geq 1$
- $\beta_0(x) \equiv 0$ for $x \geq 1$
- Implied loss compensator

$$\nu(t,(0,x]) = \alpha_0(L_t) - \alpha_0(L_t + x) + (\beta_0(L_t) - \beta_0(L_t + x))Y_t$$

- Loss given default cdf

$$\frac{\nu(t,(0,x])}{\nu(t,(0,1])} = \frac{\alpha_0(L_t) - \alpha_0(L_t + x) + (\beta_0(L_t) - \beta_0(L_t + x))Y_t}{\alpha_0(L_t) - r + \beta_0(L_t)Y_t}$$

Mixture of Exponential LGD Distributions

- $\alpha_0(x) = \gamma_0 \left(\mathrm{e}^{-a_0(x \wedge 1)} - \mathrm{e}^{-a_0}\right) + r$
- $\beta_0(x) = (1 - \gamma_0) \left(\mathrm{e}^{-b_0(x \wedge 1)} - \mathrm{e}^{-b_0}\right)$
- $\gamma_0 \in [0, 1]$ mixing parameter

Market Price of Risk

- Market price of market risk $= -\frac{\lambda}{\sigma}\sqrt{Y_t}$
- $W_t = W_t^{\mathbb{P}} - \int_0^t \frac{\lambda}{\sigma}\sqrt{Y_s}\, ds$
- Discounted (T, x)-bond $Z_t = Z(t, T, x)$

$$\frac{dZ_t}{Z_{t-}} = B(T-t, x)\lambda Y_t\, dt - B(T-t, x)\sigma\sqrt{Y_t}\, dW_t^{\mathbb{P}} + \cdots d(\mu-\nu)$$

- Factor process

$$dY_t = (\mu_0 + (\mu_1 - \lambda)Y_t)dt + \sigma\sqrt{Y_t}\, dW_t^{\mathbb{P}}$$

Outline

Data

- iTraxx Europe Data from Bank Austria
- Period: 26 Sep 2005 to 8 Jan 2009
- Stripped data: pre-default zero-coupon bonds on tranches

$$p(t,\tau,i) = \int_{(x_{i-1},x_i]} \mathrm{e}^{-\int_t^T f(t,u,x)\,du}\,dx$$

for

- Maturities $\tau = 1, 3, 5, 7, 10$
- Tranches $i = 1, \ldots, 6$

Calibration

- Method: extended Kalman filter QML (Duffee–Stanton [DS04])
- Approximate $\beta_0(x) \approx \sum_{i=1}^{6} \beta_i 1_{(0,x_i]}(x)$ where

$$\beta_i = \frac{1}{x_i - x_{i-1}} \int_{(x_{i-1},x_i]} \beta_0(x)\, dx$$

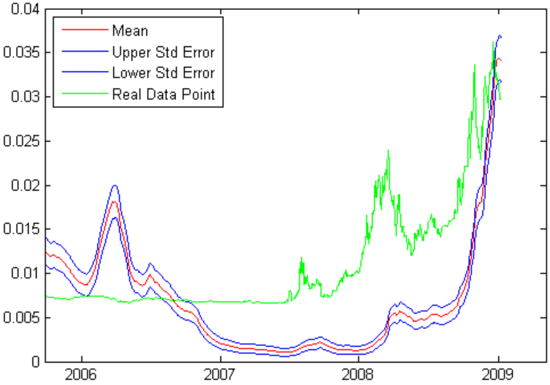
- Observation equation becomes linear in Y_t:

$$-\log p(t,\tau,i) = -\log \int_{(x_{i-1},x_i]} \mathrm{e}^{-A(\tau,x)}\, dx + B(\tau,x_i) Y_t + \epsilon(t,\tau,i)$$

- Maximize likelihood w.r.t. parameters $\mu_0, \mu_1, \sigma, \lambda, a_0, b_0, \gamma_0, h = \mathrm{Var}[\epsilon]$

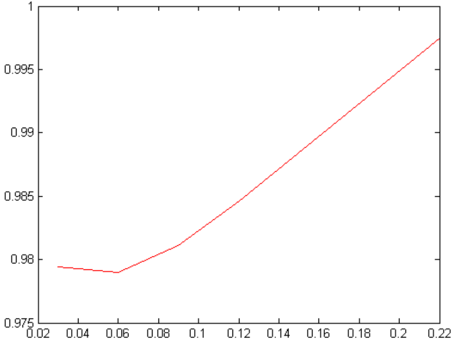
Filtered Factor Process

26 Sep 2005 to 8 Jan 2009: 819 trading days



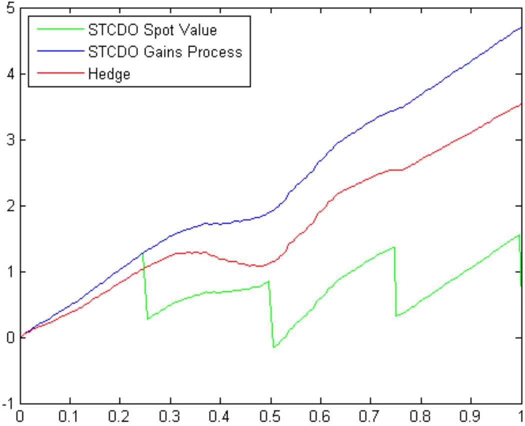
Base Correlation Skew

- Equity tranche $\mathbb{E}_{\mathbb{Q}}[L_T \wedge K] = K - \mathrm{e}^{rT} \int_{(0,K]} P(0, T, x)dx$
- $T = 3$, individual $\mathbb{Q}$-default probability $= 8\%$



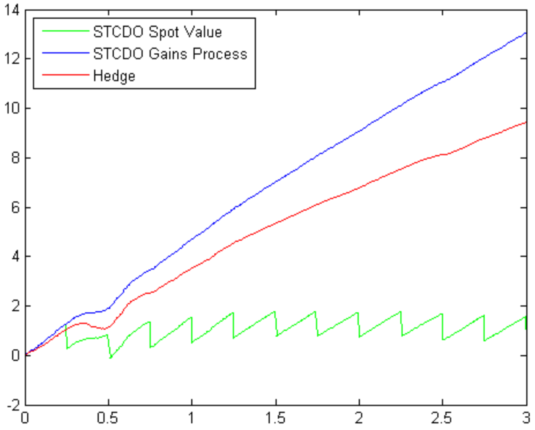
Hedging in Sample

Daily rebalancing, time horizon $= 1$ year, $3 - 6\%$ tranche



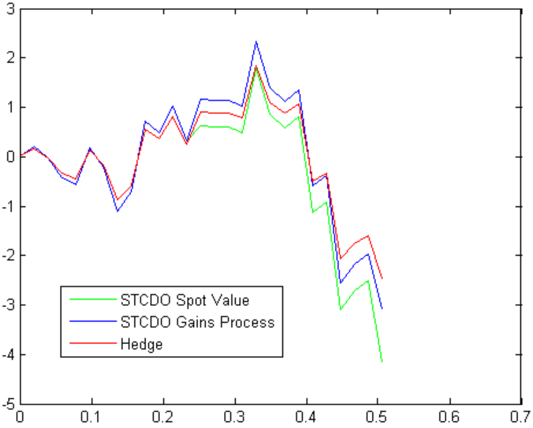
Hedging in Sample

Daily rebalancing, time horizon $= 3$ years, $3 - 6\%$ tranche



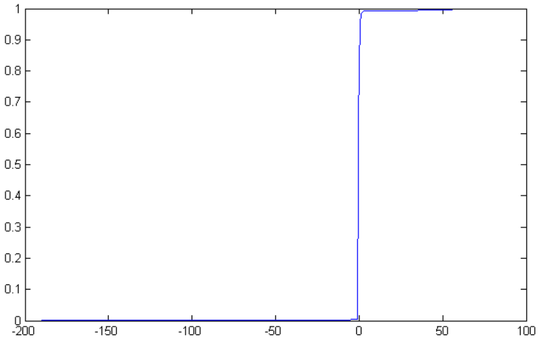
Hedging on Simulated Scenarios: A Sample

Weekly rebalancing, time horizon $= 1/2$ year, $3 - 6\%$ tranche



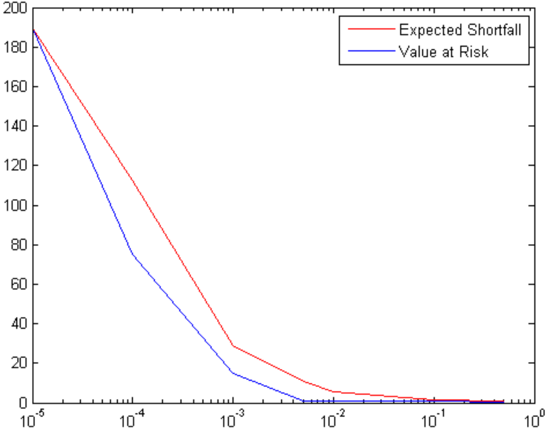
Hedging on Simulated Scenarios: P&L Distribution

Weekly rebalancing, time horizon $= 1/2$ year, $3 - 6\%$ tranche



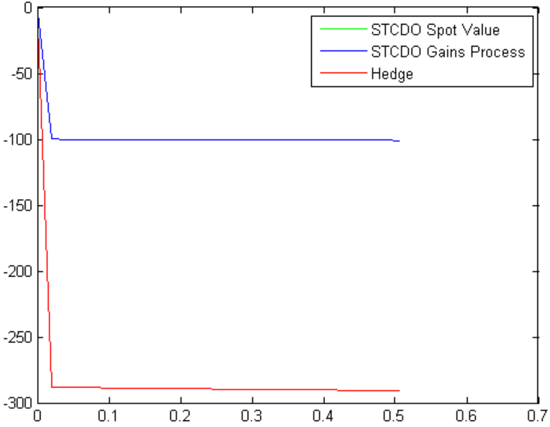
Hedging on Simulated Scenarios: P&L Tail

Weekly rebalancing, time horizon $= 1/2$ year, $3 - 6\%$ tranche



Hedging on Simulated Scenarios: Worst Case

Weekly rebalancing, time horizon $= 1/2$ year, $3 - 6\%$ tranche



Outline

Conclusion

- Top-down model for (T, x)-bonds
- Explicit variance-minimizing hedging strategy
- Simple one-factor affine model
- Captures base correlation skew
- Acceptable hedging performance

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