Equilibrium with Heterogeneous Agents

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Benchmark Case: One Agent

- CRRA representative agent with risk aversion γ;
- dividend following a geometric Brownian motion with volatility σ^D;
- then, the market price of risk is given by $\lambda = \gamma\sigma^D$;
- the stock price volatility is $\sigma^S = \sigma^D$;
- the optimal portfolio is myopic, equal to $\frac{\lambda}{\sigma^S\gamma}$.

Heterogeneous Preferences

- Dumas (1989)
- Wang (1993, 1996)
- Detemple and Murthy (1997)
- Kogan, Ross, Wang and Westerfield (2006), (2008)
- Berrada, Hugonnier and Rindisbacher (2007)
- Berrada (2008)
- Jouini and Napp (2007)
- Yan (2008)
- Bhamra and Uppal (2009)
- Weinbaum (?)

The model

- risk-free asset with $r = 0$
- one stock with terminal dividend D_T, such that

$$D_t^{-1}\, dD_t \;=\; \mu^D(D_t)\, dt \;+\; \sigma^D(D_t)\, dB_t\,;$$

- K agents, agent k is initially endowed with $\psi_k > 0$ shares of stock;
- agent k chooses $\pi_{k\,t}$, the portfolio weight at time t in the risky asset, as to maximize the expected utility

$$E\left[u_k(W_{k\,T})\right]$$

Equilibrium

- optimal terminal wealth

$$W_{k\,T} \;=\; I_k(y_k\,M) \text{ where } I = (u')^{-1}$$

with

$$E[I_k(y_k\,M)\,M] \;=\; W_{k\,0} \;=\; \psi_k\,S_0 \;=\; \psi_k\,E[DM].$$

- equilibrium SDF M solves

$$\sum_{k=1}^{K} \; I_k(y_k\,M) = \; D\,;$$

- Q is the equilibrium risk neutral measure,

$$E_t^Q[X] \;=\; \frac{E_t[MX]}{E_t[M]}.$$

Representative Agent's Utility U

- Equilibrium SDF has to satisfy

$$M = U'(D); \tag{1}$$

- aggregate risk aversion is

$$\gamma^U(x) := -\frac{x\,U''(x)}{U'(x)}.$$

Rate of Macroeconomic Fluctuations

- $$F(x) \stackrel{def}{=} \int_{x_0}^{x} \frac{1}{y\,\sigma^D(y)}\,dy\,.$$

  Then, for $A_t = F(D_t)$ we have

  $$dA_t \;=\; -C(A_t)\,dt \;+\; dB_t$$

- We call $c(D_t) = C'(A_t)$ the rate of macroeconomic fluctuations.
- Can be shown

$$c(x) = \; -x\,(\mu^D)'(x) \;+\; x\,(\sigma^D)'(x)\,\sigma^D(x)^{-1}\,\mu^D(x) \\ +\;(\sigma^D)'(x)\,\sigma^D(x)\,x \;+\; 0.5\,(\sigma^D)''(x)\,x^2\,\sigma^D(x) \quad (2)$$

Properties of the Rate of Macroeconomic Fluctuations

Proposition

- if $D_t \;=\; g(\tilde{D}_t)$ then

$$c^D(g(x)) \;=\; c^{\tilde{D}}(x).$$

- $c(D_t) \equiv b$ is constant if and only if there exists a one-to-one function g such that $D_t \;=\; g(A_t)$ where

$$dA_t \;=\; (a - bA_t)\,dt \,+\, \sigma^A dB_t.$$

Equilibrium Market Price of Risk

Theorem

$$\lambda_t \; = \; \frac{\mu_t^S \, - \, r}{\sigma_t^S}$$

is given by

$$\lambda_t \; = \; E_t^Q \left[\gamma^U(D) \, \sigma^D(D) \, e^{- \int_t^T \, c(D_s) \, ds} \right].$$

Corollary Under the equilibrium risk neutral measure, the drift of the equilibrium market price of risk is always equal to $c(D_t)$.

Discounted Volatility

- We denote by

$$\sigma^D(t,T) \stackrel{def}{=} e^{-\int_t^T c(D_s)\,ds}\,\sigma^D(D_T)$$

the discounted volatility.

- The market value of discounted volatility is defined as

$$\sigma_t^{\text{myopic}} := E_t^Q\left[\,\sigma^D(t\,,\,T)\,\right]$$

Risk Aversion and Market Price of Risk

Let

$$\gamma_k^{\inf}\ ,\ \gamma_k^{\sup}$$

denote the infinum and supremum of the relative risk aversion of agent k,

Proposition The equilibrium market price of risk satisfies

$$\min_k \gamma_k^{\inf}\ \le\ \frac{\lambda_t}{\sigma_t^{\text{myopic}}}\ \le\ \max_k \gamma_k^{\sup}.$$

Special case: CRRA agents, lognormal dividend: $\sigma_t^{\text{myopic}} \equiv \sigma$

Cyclicality of the Market Price of Risk

Proposition We have

- if $\gamma^U(x)\,\sigma^D(x)$ is countercyclical and $c(D_t)$ is procyclical, then λ_t is countercyclical;
- if $\gamma^U(x)\,\sigma^D(x)$ is increasing and $c(D_t)$ is countercyclical, then λ_t procyclical.

Equilibrium Stock Volatility

Theorem The equilibrium stock price volatility is given by

$$\sigma_t^S \;=\; \sigma_t^{\text{myopic}} \;+\; \frac{1}{E_t^Q[D]}\,\mathrm{Cov}_t^Q\left([1-\gamma^U(D)]\,\sigma^D(t\,,\,T)\,,\,D\right)\,. \tag{3}$$

Furthermore, S_t is always procyclical and $\sigma_t^S > 0$ almost surely.
With constant $\gamma^U, \sigma^D, c = b$,

$$\sigma^S = e^{-b(T-t)}\sigma^D$$

Excess Volatility

Proposition

- if the relative risk aversion $\gamma^U(x) \geq 1$ is decreasing, the volatility $\sigma^D(D)$ is countercyclical and the rate $c(D_s)$ is procyclical, then

$$\sigma_t^S \ \geq \ \sigma_t^{\text{myopic}}\,.$$

- if the relative risk aversion $\gamma^U \geq 1$ is increasing, the volatility $\sigma^D(D)$ is procyclical and the rate $c(D_s)$ is counter-cyclical, then

$$\sigma_t^S \ \leq \ \sigma_t^{\text{myopic}}\,.$$

Example: The Gaussian Case

$$d\log(D_t) \;=\; (a - b\log(D_t))dt \,+\, \sigma\, dB_t.$$

In that case, $\sigma^D = \sigma,\; c \;=\; b,$

Proposition Suppose that γ^U is decreasing. Then,

- market price of risk is counter-cyclical and satisfies

$$\min_k \gamma_k^{\inf} \;\le\; \frac{\lambda_t}{e^{b(t-T)}\,\sigma} \;\le\; \max_k \gamma_k^{\sup}\,;$$

- price volatility is larger than the discounted (or, appreciated, if $b < 0$) volatility,

$$\sigma_t^S \;>\; e^{b\,(t\,-\,T)}\,\sigma^D\,.$$

Special case: CRRA agents, $b = 0$: $\sigma \le \sigma_t^S \le \sigma(1 + \max\gamma_k - \min\gamma_k)$.

Risk-Return Tradeoff

$$\frac{\lambda_t}{\sigma_t^S} \;=\; \gamma\,\frac{\sigma_t^{\mathrm{myopic}}}{\sigma_t^{\mathrm{myopic}}\,+\,\sigma_t^{\mathrm{nonmyopic}}}\,.$$

Prediction: variation of the risk-return tradeoff is explained by fluctuations in non-myopic (excess) volatility

Optimal Portfolios and the Log-Optimal Portfolio

Log-optimal portfolio is

$$\pi_{\log t} \;=\; \frac{\lambda_t}{\sigma_t^S}.$$

Proposition If $\gamma_k(x) \geq 1$ for all x, then

$$\pi_{kt} \;\leq\; \pi_{\log t}$$

and the inequality reverses if, for all x, $\gamma_k(x) \;\leq\; 1$.

Monotonicity in Risk Aversion

Definition (Ross (1981)) Agent k is more risk averse than agent j in the sense of Ross if

$$\inf_x \gamma_k(x) \;\geq\; \sup_x \gamma_j(x).$$

In this case we write $\gamma_k \;\geq_R\; \gamma_j$.

Proposition

- suppose that $\gamma^U(x)\,\sigma^D(x)$ is decreasing and c is procyclical. Then, $\gamma_k \;\geq_R\; \gamma_j \;\geq\; 1$ implies

$$\pi_{kt} \;\leq\; \pi_{jt}\,;$$

- suppose that $\gamma^U(x)\,\sigma^D(x)$ is increasing and c is countercyclical. Then, $1 \;\geq\; \gamma_k \;\geq_R\; \gamma_j$ implies

$$\pi_{kt} \;\leq\; \pi_{jt}\,.$$

Myopic Portfolio

- $$U_{k\,t}(x) \;=\; \sup_{\pi} E_t\left[\, u_k(W_{kT}) \,|\, W_{kt} \,=\, x \,\right]$$

  denotes the value function of agent k;

- $$\gamma_{k\,t} \;=\; -\,\frac{x\, U''_{k\,t}(W_t)}{U'_{k\,t}(W_t)}.$$

  denotes the effective relative risk aversion of agent k at time t;

- Define myopic portfolio as

$$\pi_{k\,t}^{\text{myopic}} \;\stackrel{def}{=}\; \frac{\lambda_t}{\gamma_{k\,t}\, \sigma_t^S}\,.$$

Hedging Portfolio

We will denote

$$\pi_{k\,t}^{\text{hedging}} \; = \; \pi_{k\,t} \; - \; \pi_{k\,t}^{\text{myopic}}$$

$$\lambda(t\,,\,\tau) \;\stackrel{def}{=}\; e^{-\int_t^\tau \, c(D_s)\,ds}\,\lambda_\tau$$

Theorem We have

$$\pi_{k\,t}^{\text{hedging}} \; = \; - \, \frac{1}{\sigma_t^S\,W_{k\,t}}\,\text{Cov}_t^Q\left(\lambda(t,T)\,,\,W_{kT} \; - \; R(W_{kT})\;\right)$$

$$\pi_{k\,t}^{\text{myopic}}W_{kt} \;\stackrel{def}{=}\; \frac{\lambda_t}{\sigma_t^S}R_{kt} = \frac{\lambda_t}{\sigma_t^S}E_t^Q[R_{kT}]\,.$$

where $R(x) = x/\gamma(x)$ is absolute risk tolerance.

The Sign of the Hedging Portfolio

$$P_k(x) \; := \; -\frac{u'''(x)}{u''(x)} \; , \; r_k(x) \; := \; x\gamma_k^{-1}(x)$$

denote the absolute prudence and absolute risk tolerance of agent k.

Proposition

- If $\gamma^U(x)\,\sigma^D(x)$ is decreasing and c is procyclical, then

$$\begin{aligned} \pi_{kt}^{\text{hedging}} \; &\geq \; 0 \qquad if \quad \sup_x \left(P_k(x)\,R_k(x)\right) \; \leq \; 2 \\ \pi_{kt}^{\text{hedging}} \; &\leq \; 0 \qquad if \quad \inf_x \left(P_k(x)\,R_k(x)\right) \; \geq \; 2 \end{aligned} \tag{4}$$

Reason why:

- $$\frac{d}{dx}\,R_k(x) \; = \; -1 \; + \; P_k(x)\,R_k(x).$$

Increasing Relative Risk Aversion

Corollary Suppose that $\gamma_k(x) \geq 1$ and is increasing. Then,

- if $\gamma^U(x)\,\sigma^D(x)$ is decreasing and c is procyclical, then

$$\pi_{kt}^{\text{hedging}} \geq 0\,; \tag{5}$$

- if $\gamma^U(x)\,\sigma^D(x)$ is increasing and c is countercyclical, then

$$\pi_{kt}^{\text{hedging}} \leq 0\,. \tag{6}$$

Constant Relative Risk Aversion

Corollary Suppose that $\gamma_k = \text{const}$. Then,

(1) if $\gamma^U(x)\,\sigma^D(x)$ is decreasing and c is procyclical, then

$$\begin{aligned} \pi_{kt}^{\text{hedging}} &\geq 0 \qquad if\ \gamma_k \geq 1 \\ \pi_{kt}^{\text{hedging}} &\leq 0 \qquad if\ \gamma_k \leq 1\,; \end{aligned} \tag{7}$$

(2) if $\gamma^U(x)\,\sigma^D(x)$ is increasing and c is countercyclical, then

$$\begin{aligned} \pi_{kt}^{\text{hedging}} &\geq 0 \qquad if\ \gamma_k \leq 1 \\ \pi_{kt}^{\text{hedging}} &\leq 0 \qquad if\ \gamma_k \geq 1\,. \end{aligned} \tag{8}$$

Conclusions I.

- the market price of risk is determined by the aggregate (relative) risk aversion multiplied by dividend volatility discounted at the rate we call the *“rate of macroeconomic fluctuations”*;
- the stock price volatility = *excess* component + *fundamental* component. The fundamental component is given by the market value of discounted dividend volatility, myopic volatility;
- excess volatility is given by a volatility risk premium, whose sign is determined by the co-movement of the dividend with the market price of risk;

Conclusions II.

- excess volatility is positive when risk aversion and discounted volatility are counter-cyclical;
- the non-myopic (hedging) component of an agent's portfolio is given by a portfolio risk premium, whose sign is determined by the co-movement of agent's wealth and risk tolerance with the market price of risk;
- when market price of risk is counter-cyclical, hedging component is positive for CRRA agents with $\gamma \geq 1$.

Survival of CRRA agents

Theorem The agent 0 whose risk aversion is closest to 1 dominates in the long run:

$$\lim_{T \to \infty} \frac{W_{kT}}{W_{0T}} \ = \ 0$$

for all $k \neq 0$.

Relative Extinction

Definition (KRWW (2006)): agent i experiences extinction relative to agent j if

$$\lim_{t \to \infty} \frac{W_{iT}}{W_{jT}} \; = \; 0$$

Theorem. Even if agent i experiences extinction relative to agent j, adding a third agent k to the economy may reverse the situation and force the agent j to experience extinction relative to agent i.

Global Bounds for the Price

Proposition

$$e^{r(t-T)} \frac{E_t[D^{1-\gamma_{\max}}]}{E_t[D^{-\gamma_{\max}}]} \;\le\; S_t \;\le\; e^{r(t-T)} \frac{E_t[D^{1-\gamma_{\min}}]}{E_t[D^{-\gamma_{\min}}]}$$

Related to:

- bubbles and crashes (Cao and Ou-Yang (2005));
- Harrison and Kreps (1978).

Large population limit

Assumption. Risk aversions densely cover an interval $[1, \Gamma]$.

Long run dynamics

Definition. Given a random process X_t, $t \in [0, T]$, we define

$$X(\lambda) = \lim_{T\to\infty} X_{\lambda T}$$

for $\lambda \in (0,1)$.

Long run drift and volatility

Theorem

$$\mu(\lambda) \;=\; \begin{cases} r \,+\, (1+\lambda^{-1})^2\,\sigma^2\,, & \lambda \,\ge (\Gamma-1)^{-1} \\ r \,+\, \Gamma^2\,\sigma^2\,, & \lambda \,<\, (\Gamma-1)^{-1} \end{cases}$$

and (9)

$$\sigma(\lambda) \;=\; \begin{cases} \sigma\,(1 \,+\, \lambda^{-1})\,, & \lambda \,\ge (\Gamma-1)^{-1} \\ \sigma\,\Gamma\,, & \lambda \,<\, (\Gamma-1)^{-1} \end{cases}$$

The special role of risk aversion two

If $\Gamma \geq 2$ then, for $t \approx T$, volatility is two times larger,

$$\sigma_t \approx 2\sigma .$$

The market price of risk

$$\frac{\mu_t - r}{\sigma_t} \approx 2\,\sigma$$

is determined by the agent with risk aversion 2
and not by the log agent!

Volatility and MPR are Decreasing

Corollary In the limit $T \to \infty$, the instantaneous drift, the volatility and the market price of risk of the stock are monotone decreasing in $t = \lambda T$.

Long run myopic portfolios

Proposition

$$\pi_\gamma^{\mathrm{myopic}}(\lambda) \;=\; \frac{1}{\gamma}$$

is independent of λ.

Long run hedging portfolios

Theorem. We have

- if $\lambda > (\Gamma - 1)^{-1}$ then

$$\pi_\gamma(\lambda) \; = \; \gamma^{-1} + \frac{\gamma - 1}{(\lambda + 1)\,\gamma\,(1 + \lambda\,(\gamma - 1))} \; ;$$

- if $\lambda < (\Gamma - 1)^{-1}$ then

$$\pi_\gamma(\lambda) \; = \; \gamma^{-1} + (\gamma - 1)\,\frac{(\Gamma - 1)\,(1 + \lambda\,(\gamma - 1)) \; - \; (\gamma - 1)}{\Gamma\,\gamma\,(1 + \lambda\,(\gamma - 1))}.$$

Monotonicity properties

Proposition. Let $\lambda > (\Gamma - 1)^{-1}$. Then,

- the hedging portfolio
$$\pi_\gamma^{\text{hedging}}(\lambda)$$
is monotone decreasing in λ for each fixed γ;
- for each fixed λ, $\pi_\gamma^{\text{hedging}}(\lambda)$ is monotone increasing in γ for
$$\gamma < 1 + \lambda^{-1/2}$$
and is monotone decreasing for $\gamma > 1 + \lambda^{-1/2}$.

Conclusions

- With more then two agents, agents impact relative extinction of each other;
- Long run volatility is two times larger than dividend volatility and the long run market price of risk is determined by the agent with risk aversion **two**;
- Hedging demand never vanishes and may exhibit unexpected patterns in terms of risk aversion;
- Close to $t = T$, agent with risk aversion **two** has the highest hedging demand.

Intertemporal Consumption

Joint with **E. Jouini** and **C. Napp**.
Heterogeneous beliefs, and CRRA agents:

$$dD_t = \mu_i D_t dt + \sigma D_t dW_t^i$$

$$E^{P^i}\left[\int_0^\infty e^{-\rho_i t} u_i\left(c_t\right) dt\right]$$

Denote $\delta_i = (\mu_i - \mu)/\sigma$ and introduce Survival Index:

$$-\rho_i - \gamma_i(\mu_i - \frac{\sigma^2}{2}) - \frac{1}{2}\delta_i^2$$

Results: SDF and market price of risk

- In the long run, SDF is determined by the agent with highest survival index, who also dominates in terms of consumption shares.
- In very bad (good) states the SDF is determined by the agent associated with the highest (lowest) market price of risk, who also dominates in terms of consumption shares.
- Market price of risk is a (moving) weighted average of “individual” market prices of risk. This is not the case for the short rate. Asymptotics are as above.
- Market price of risk is always decreasing in aggregate wealth.

Results: short rate and volatility

- The short rate converges to the one corresponding to the highest survival index, while the long-term yield converges to the lowest individual short rate. At other horizons, other agents determine the asymptotic driving marginal discount rate, in accordance with Preferred Habitat Theory.
- Even in the long-run, the price of an asset is not necessarily equal to the price corresponding to the highest survival index.
- The stock volatility is given by

$$\sigma_S\left(t\right) = \sigma + \frac{E_t\left[\int_t^\infty (\theta_t - \theta_\tau) M_\tau D_\tau d\tau\right]}{E_t\left[\int_t^\infty M_\tau D_\tau \, d\tau\right]}.$$

converges to σ, and satisfies

$$\sigma + \min_i \, \theta_i - \max_i \theta_i \ \leq \ \sigma_t^S \leq \sigma + \max_i \theta_i - \min_i \theta_i$$

- The asymptotic stock price long run return rate is not necessarily equal to the one corresponding to the surviving agent.

Results: Optimal portfolios

The optimal portfolio is given by, with $b_i = 1/\gamma_i$,

$$\sigma_t \pi_{it} = \theta_t + \frac{E_t\left[\int_t^\infty (b_i\delta_i + (b_i - 1)\theta_\tau) M_\tau c_{i\tau} d\tau\right]}{E_t\left[\int_t^\infty M_\tau c_{i\tau} d\tau\right]}$$

In particular,

$$\min_j \theta_j + \min_j (b_i\delta_i + (b_i - 1)\theta_j) \leq \sigma_t \pi_{it} \leq \max_j \theta_j + \max_j (b_i\delta_i + (b_i - 1)\theta_j)$$

If we further assume that $\gamma_i > 1$, for all i, then, with I_K being the highest survival index,

$$\lim_{t\to\infty} \pi_{it} = \frac{\delta_i + \theta_{I_K}}{\sigma\gamma_i}.$$

Thank you for your patience :))